COEFFICIENT INEQUALITY FOR A SUBCLASS OF REGULAR p-VALENT FUNCTIONS

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Abstract: Here we describe some classes of analytic functions and its subclasses by which we will be obtaining sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$. In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \le 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \le 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \le 2$ and $|a_3| \le 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 3 - 4\mu, if\ \mu \le 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \le \mu \le 1; \\ 4\mu - 3, if \mu \ge 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

$\in \mathcal{A}$ and satisfying the condition

$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.3)

We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$ and satisfying the condition

$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}.$ (1.4)

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$Re\left(\frac{zf\prime(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.5)

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in \mathbb{E}\right\}$ (1.6)

$\mathcal{K}(A,B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))\prime}{f\prime(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in \mathbb{E}\right\}$ (1.7)

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as $\left\{\mathbf{f(z)} \in \mathcal{A}; \frac{\mathbf{z(zf'(z))'}}{\mathbf{z(f'(z))}} \prec \mathbf{p}\frac{\mathbf{1+w(z)}}{\mathbf{1-w(z)}}; \mathbf{z} \in \mathbb{E}\right\}$ and we will denote this class as $f(z) \in KS_p^*]$,

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z)); z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.8)

It is known that $|d_1| \le 1, |d_2| \le 1 - |d_1|^2.$ (1.9)

Preliminary Lemmas: For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + --- \quad (2.1)$$

Main Results

THEOREM 3.1: Let $f(z) \in KS_p^*$, then

$$|a_{p+2}-\mu a_{p+1}^2| \le \begin{cases} \left[\frac{p^2(2p+1)}{p+2}-\frac{\mu 4p^4}{(p+1)^2}\right];\ if\ \mu \le \frac{(p+1)^2}{2p(p+2)} & (3.1) \\ \frac{p^2}{(p+2)};\ if\ \frac{(p+1)^2}{2p(p+2)} \le \mu \le \frac{(p+1)^3}{2p^2(p+2)} & (3.2) \\ \left[\frac{4p^4\mu}{(p+1)^2}-\frac{p^2(2p+1)}{p+2}\right];\ if\ \mu \ge \frac{(p+1)^3}{2p^2(p+2)} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $f(z) \in KS_p^*$, , we have

$$\frac{\mathbf{z(zf'(z))'}}{\mathbf{z(f'(z))}} = p\frac{1+w(z)}{1-w(z)} \ ;\ w(z) \in \mathcal{U}. \qquad (3.4)$$

Expanding the series (3.4), we get

$$\{p^2z^p + (p+1)^2a_{p+1}z^{p+1} + (p+2)^2a_{p+2}z^{p+2} - -\} = p(1 + 2c_1z + 2(c_2+c_1^2)z^2 + - -) \qquad (3.5)$$

Identifying terms in (3.5), we get

$$a_{p+1} = \frac{2p^2}{(p+1)}c_1 \qquad (3.6)$$

$$a_{p+2} = \frac{p^2}{(p+2)}c_2 + \frac{[p^2(2p+1))]}{(p+2)}c_1{}^2. \qquad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_{p+2}-\mu a_{p+1}^2 = \frac{(p)^2}{(p+2)}c_2 + \left[\frac{2p^3+p^2}{(p+2)}-\frac{4p^4\mu}{(p+1)^2}\right]c_1^2 \qquad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_{p+2}-\mu a_{p+1}^2| \le \frac{(p)^2}{(p+2)}|c_2| + \left|\frac{p^2(2p+1)}{(p+2)}-\frac{4p^4\mu}{(p+1)^2}\right||c_1^2|. \qquad (3.9)$$

Now we get,

$$|a_{p+2}-\mu a_{p+1}^2| \le \frac{(p)^2}{(p+2)}(1-|c_1|^2) + \left|\frac{p^2(2p+1)}{(p+2)}-\frac{4p^4\mu}{(p+1)^2}\right||c_1^2|$$

$$= \frac{(p)^2}{(p+2)} + \left[\left|\frac{p^2(2p+1)}{(p+2)}-\frac{4p^4\mu}{(p+1)^2}\right| - \frac{(p)^2}{(p+2)}\right]|c_1|^2. \qquad (3.10)$$

Case I: $\mu \le \frac{(p+1)^2(2p+1)}{4p^2(p+2)}$. (3.10) can be rewritten as

$$|a_{p+2}-\mu a_{p+1}^2| \le \frac{(p)^2}{(p+2)} + \left[\frac{2(p)^3}{(p+2)}-\frac{4p^4\mu}{(p+1)^2}\right]|c_1|^2. \qquad (3.11)$$

Subcase I (a): $\mu \le \frac{(p+1)^2}{2p(p+2)}$. Using (1.11), (3.11) becomes

$$|a_{p+2}-\mu a_{p+1}^2| \le \left[\frac{p^2(2p+1)}{(p+2)}-\frac{4p^4\mu}{(p+1)^2}\right] \qquad (3.12)$$

Subcase I (b): $\mu \ge \frac{(p+1)^2}{2p(p+2)}$. We obtain from (3.11)

$$|a_{p+2}-\mu a_{p+1}^2| \le \frac{(p)^2}{(p+2)}. \qquad (3.13)$$

Case II: $\mu \ge \frac{(p+1)^2(2p+1)}{4p^2(p+2)}$

Preceding as in case I, we get

$$|a_{p+2}-\mu a_{p+1}^2| \le \frac{(p)^2}{(p+2)} + \left[\frac{4p^4\mu}{(p+1)^2}-\frac{p^2(p+1)}{(p+2)}\right]|c_1|^2. \qquad (3.14)$$

Subcase II (a): $\mu \le \frac{(p+1)^3}{2p^2(p+2)}$

(3.14) takes the form $|a_{p+2}-\mu a_{p+1}^2| \le \frac{(p)^2}{(p+2)}$ (3.15)

$$|a_{p+2}-\mu a_{p+1}^2| \le \frac{(p)^2}{(p+2)} \ ;\ if\ \frac{p^2(2p+1)}{(p+2)} \le \mu \le \frac{(p+1)^3}{2p^2(p+2)} \qquad (3.16)$$

Subcase II (b): $\mu \ge \frac{(p+1)^3}{2p^2(p+2)}$

Preceding as in subcase I (a), we get

$$|a_{p+2}-\mu a_{p+1}^2| \le \left[\frac{4p^4\mu}{(p+1)^2}-\frac{p^2(2p+1)}{(p+2)}\right] \qquad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = \frac{z^p}{p} + \frac{p\,z^{p+1}}{p+1} + \frac{p(p-1)}{2!}\ \frac{z^{p+2}}{p+2} + - - - - - -$$

Extremal function for (3.2) is defined by

$$f_2(z) = \frac{z^p}{p} + \frac{p\,z^{p+2}}{p+2} + \frac{p(p-1)}{2!}\ \frac{z^{p+4}}{p+4} + - - - - -.$$

Corollary 3.2: Putting $p = 1$ in the theorem, we get

$$|a_{p+2}-\mu a_{p+1}^2| \le \begin{cases} 1-\mu, if\mu \le \frac{2}{3}; \\ \frac{1}{3} if \frac{2}{3} \le \mu \le \frac{4}{3}; \\ \mu - 1, if\ \mu \ge \frac{4}{3} \end{cases}$$

These are the required results of class KS*.

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