A SIMPLE PROOF OF JACOBI'S FOUR-SQUARE THEOREM AND ITS APPLICATIONS

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Abstract: A celebrated result, due to Jacobi, says that the number of representations of the positive integer n as a sum of four squares is equal to eight times the sum of the divisors of n which are not divisible by 4. We give a new and simple proof of this result which depends only on Jacobi's triple product identity.

Key words: Ring of Gaussians Integers, Quaternion, Commutatively, left ideal.

Introduction: Pierre de Fermat's theorem on sums of two squares states that a prime p of the form 4n+1 is expressible as $p = x^2 + y^2$, with x and y integers.

For example, the primes 5, 13, 17, 29, 37 and 41 are all congruent to 1 modulo 4, and they can be expressed as sums of two squares in the following ways:

For example

$$5 = 1^2+2^2, \quad 13 = 2^2+3^2, \quad 17 = 1^2+4^2, \quad 29 = 2^2+5^2, \quad 37 = 1^2+6^2, \quad 41 = 4^2+5^2.$$

Now we have to prove the interesting theorem that theorem that every positive integer is a sum of four squares. For which first we have to do some preparatory work as under.

Quaternion: On October 16th,1843, while walking with his wife to a meeting of the royal society of Dublin, Hamilton discovered a 4-dimensional division algebra called Quaternion. The quaternions are H = $\{a1 + bi + cj + dk \ / \ a, b, c, d \in R\}$ s. t. i, j , k , are square roots of -1.

Implies that i j = k = - ji, ,jk = i = - k j ,
k i = j = -i k

Or can be defined that $i^2 = j^2 = k^2$ = i j k = - 1

Quaternion H is equal to R^4, a four dimensional vector space over reals and has three operations i.e. Addition, Scalar multiplication and Quaternions multiplication. The basis element 1 will be the identity element of H. Elements of H are usually written as a+bi+ cj +dk , suppressing the basis element 1.

Let Q be the division ring of real quaternions. Then we define adjoint of x as:

Adjoint of x: For x= $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ in Q. The ad joint of x denoted by x^* , is defined by

$x^* = \alpha_0 - \alpha_1 i - \alpha_2$ j - $\alpha_3 k$.

Lemma I: The adjoint of x for x $\in Q$, where Q is a Quaternion satisfies the following:

1. $x^{**} = x$
2. $(\alpha x + \beta y)^* = \alpha x^* + \beta y^*$
3. $(xy)^* = y^* x^*$ for all x , y in Q $and\ \alpha, \beta\ in\ R$

Proof: If x= $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ then x^* = $\alpha_0 - \alpha_1 i - \alpha_2$ j - $\alpha_3 k$, hence x^{**} = $(x^*)^*$= $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ which proves part 1.

Let x= $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ and y= $\beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$ be in Q and α, β be real numbers. Then αx+ βy = $(\alpha\alpha_0 + \beta\beta_0) + (\alpha\alpha_1 + \beta\beta_1)i + (\alpha\alpha_2 + \beta\beta_2)j + (\alpha\alpha_3 + \beta\beta_3)k$ therefore by definition of the Adjoint we have (αx + βy) *=($\alpha\alpha_0 + \beta\beta_0) - (\alpha\alpha_1 + \beta\beta_1)i - (\alpha\alpha_2 + \beta\beta_2)j - (\alpha\alpha_3 + \beta\beta_3)k$=α($\alpha_0 - \alpha_1 i - \alpha_2$ j - $\alpha_3 k$)+β ($\beta_0 - \beta_1 i - \beta_2 j - \beta_3 k$)=α$x^*$+β$Y^*$ which proves part 2.

In view of part 2 , to prove part 3 it is suffices to prove it for basis vectors of Q over reals . We prove it for the basis elements 1, i, j, k. As ij=k then $(ij)^*$= k^*= -k =j i = (-j)(-i)= $(j)^*$ $(i)^*$. Similarly $(ik)^*$= $(k)^*$ $(i)^*$ and $(jk)^*$ = $(k)^*$ $(j)^*$. Also $(i^2)^*$= $(-1)^* = -1 =$ $(i^*)^2$, and similarly for j and k. As part 3 is true for basis elements then in the light of part 2, part 3 is true for all linear combinations of the basis elements with real coefficients; hence part 3 is true for all x and y in Q.

Definition: Let xϵQ then the **Norm** of x, denoted by N(x), and is defined by N(x) = xx^*. That is if x=$\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ then N(x) = xx^*=$(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)$ ($\alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$) =α_0^2 +α_1^2+$\alpha_2^2 + \alpha_3^2$.From the above definition we have N(0)=0 and N(x) is a positive number for x$\neq$ 0.

Lemma II: For x and y in Q, N (xy) =N(x) N(y).

Proof: From the above definition of norm of x, we have N (xy) = (xy) $(xy)^*$ and by part 3 of lemma I , $(xy)^* = y^* x^*$

We have N (xy) =xy$y^*$$x^*$. But yy^*= N(y) which is a real number it must commute with x^*. Consequently N (xy) = $x(yy^*)x^*$=$(xx^*)(yy^*)$=N(x) N(y).

Lemma III: (**Lagrange's Identity**): If $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and $\beta_0, \beta_1, \beta_2, \beta_3$ are real numbers. then

$(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)(\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2) = (\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3)^2 + (\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_0\beta_2 - \alpha_1\beta_3 + \alpha_2\beta_0 + \alpha_3\beta_1)^2 + (\alpha_0\beta_3 + \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_3\beta_0)^2$.

Proof: As in Lemma II we prove that N (xy) =N(x) N(y) where x= $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ and y=$\beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$ be in Q and α, β be real numbers. The proof is obvious by multiplying and comparing the respective terms. Infect Lagrange's identity says that the sum of four squares times the sum of four squares is again in a very specific way the sum of four squares .

Now we define **Hurwitz ring** of integral quaternion. Let ζ = $\frac{1}{2}(1 + i + j + k)$ and let H= $\{ m_0\zeta + m_1 i + m_2 j + m_3 k / m_0, m_1, m_2, m_3 \text{ are integers}\}$

Lemma IV: Let H be a sub ring of Q where Q is a Quaternion, and if $x \in H$ then $x^* \in H$ and N(x) i.e. Norm of x is a positive integer for every non- zero $x \varepsilon H$.

Proof. The proof of the lemma is obvious.

Lemma V: Let H be a sub ring over Q, and if a and b $\in H$ with b≠0. Then thre exit two elements c and d in H such that a= cb+ d and N (d) < N (b).

Proof: In order to prove this lemma we first prove It for a special case such that let a is an arbitrary element of H and b is a positive integer. Suppose that a=$t_0\zeta + t_1 i + t_2 j + t_3 k$ *where* t_0, t_1, t_2, t_3 *are integers* and b =n where n is a positive integer. Let c=$x_0\zeta + x_1 i + x_2 j + x_3 k$ *where* x_0, x_1, x_2, x_3 *are integers* yet to be determined. We want to choose them in such a manner as to force N (a-cn) <N (n) = n^2. But

a-cn= $(t_0(\frac{1+i+j+k}{2}) + t_1 i + t_2 j + t_3 k)$- $n\, x_0(\frac{1+i+j+k}{2})$- $n x_1 i - n x_2 j - n x_3 k$=$\frac{1}{2}(t_0 - n x_0)$+$\frac{1}{2}(t_0 + 2t_1 - n(t_0 + 2x_1))i$+$\frac{1}{2}(t_0 + 2t_2 - n(t_0 + 2x_2))j$+$\frac{1}{2}(t_0 + 2t_3 - n(t_0 + 2x_3))k$.

If we could choose the integers x_0, x_1, x_2, x_3 in such a way as to make $|t_0 - n x_0| \le \frac{1}{2}n$, $|t_0 + 2t_1 - n(t_0 + 2x_1)| \le n$, $|t_0 + 2t_2 - n(t_0 + 2x_2)| \le n$ *and* $|t_0 + 2t_3 - n(t_0 + 2x_3)| \le n$

Then we have N(a-cn) =$\frac{(t_0 - n x_0)^2}{4}$+$\frac{(t_0 + 2t_1 - n(t_0 + 2x_1))^2}{4}$+....

$\le \frac{1}{16}n^2 + \frac{1}{4}n^2 + \frac{1}{4}n^2 + \frac{1}{4}n^2 \le n^2 = N(n)$.

Now further we have to say that

1. There is an integer x_0 such that $t_0 = x_0 n + r$ where $\frac{-1}{2}n \le r \le \frac{1}{2}n$; for this x_0, $|t_0 - n x_0| = |r| \le \frac{1}{2}n$.
2. There is an integer k such that $t_0 + 2t_1$= k n+r and 0≤ $r \le n$.If k-t_0 is even , put $2x_1$= k-t_0; then $t_0 + 2t_1$= $(2x_1 + t_0)n + r$ *and* $|t_0 + 2t_1 - (2x_1 + t_0)n| = r < n$. If on the other hand k-t_0 is odd $2x_1$= k-t_0+1; thus $t_0 + 2t_1$= $(2x_1 + t_0 - 1)n + r$ =$(2x_1 + t_0)n$+r-n, whence $|t_0 + 2t_1 - (2x_1 + t_0)n|$=|r-n|≤ n since 0<r<n. Therefore we can find an integer x_1 satisfing $|t_0 + 2t_1 - (2x_1 + t_0)n| \le n$.
3. As in part 2 we can find an integer x_2 *and* x_3 which satisfy $|t_0 + 2t_2 - (2x_2 + t_0)n| \le n$ *and* $|t_0 + 2t_3 - (2x_3 + t_0)n| \le n$, *respectively*.

Now let a is an arbitrary element of H and b be a positive integer then by lemma IV b b* = n (say) be positive integer, then there exist ac ε H such that a b* =c n+ d_1 where N(d_1) <N(n). Thus (ab* -cn) <N(n), but b b* = n hence we get N(ab*-cbb*)< N(n) and so N((a-cb)b*)< N(n)=N(bb*). By lemma 2 it reduces to N(a-cb)N(b*) <N(b) N(b*). Since N(b*) > 0 we get N(a-cb)<N(b). By putting d= a-cb we have a=cb+d where N(d)<N(b). Which proves the lemma?

Lemma VI : : If L be a left ideal of H where H be a sub ring of Q . Then there exist an element u∈ L such that every element in L is a left- multiple of u i.e. if $u \in L$ then *there is* $x \in L$ is of the form x = r u where $r \in H$.

Proof: If L= (0) then is nothing to prove only we can take u=0.

Let L be a non-zero space. By lemma IV we know that norm of a non- zero element is a positive integer. Let u≠ 0 be in L whose norm is minimal over non zero elements of L. Let xϵL then by lemma V x=cu+d where N(d)< N(u). However d is in L because both x and u therefore cu is also in L which is the left ideal of H. Thus N(d)=0 if d=0 and we prove that x=cu the required result.

Lemma VII: If $a \in H$, where H be a sub ring of Quaternion Q. Then $a^{-1} \in H$ iff N(a) = 1.

Proof: If a and a^{-1} both are in H then by lemma IV both N(a) and N(a^{-1}) are positive integers. However we know that aa^{-1}=1 Hence by lemma II we have N(a) N(a^{-1})=N(aa^{-1}) =N(1)= 1. Which proves the result? Otherwise if a∈ H and N(a)=1 then aa*= N(a)=1 then a^{-1}= a*. Then by lemma IV we have $a \in H$ and a*$\in H$, so that a^{-1} =a* is also in H.

Lagranges Four Square Theorem: Every positive integer can be expressed as the sum of squares of four integers.

Proof: Let n be a positive integer we have to prove that n=$x_0^2+x_1^2+x_2^2+x_3^2$ for $four\ integers\ x_0, x_{1,} x_{2,} x_3$. As every integer is the product of prime numbers, if every prime number were realizable as a sum of four squares , then in the light of Lagrange's identity every integer will be expressible as a sum of four squares . Thus we have reduced the problem to prime numbers only. As the prime number 2 can be written as $1^2+1^2+0^2+0^2$ as the sum of four squares.

Let n be a prime number and denote it by p as usual. Consider the quaternion W_p over J_p , the integers mod p; I.e. $W_p=\{\alpha_0+\alpha_1 i+\alpha_2 j+\alpha_3 k\ /\ \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in J_p\}$. Then W_p is a finite ring, moreover since p≠ 2 it is not commutative for ij=-ji ≠ji. Thus by **Wedderburn's theorem** it cannot be a division ring and it must have a left ideal which is neither (o) nor W_p.

Now we define two sided ideal V in H as

V=$\{x_0\zeta+x_1 i+x_2 j+ x_3 k\ \ /where\ \ p\ divide\ \ all\ of\ x_0, x_1, x_{2,} x_3\}$

cannot be a maximal left-ideal of H, since H/V the quotient space of H in V is isomorphic to W_p . (Because of the fact that if V is a maximal left ideal of H ,H/V and so W_p, would have no left ideal other than (o) and H/V.)

Thus there is another left ideal L of H. such that L≠H and L≠ V and $L \supset V$. Then by lemma VI there is an element u$\in L$. Such that every element in L is a left multiple of u . Since p$\in$ V and p$\in L$ hence p=cu for some c $\in$H. Since u $\notin$V, c cannot have an inverse in H , otherwise u=c^{-1}p would be in V . Thus N(c) > 1 by lemma VII . Since L ≠ H, u cannot have an inverse in H, whence N (u) > 1. As p=cu, p^2 = N(p) = N(cu) = N(c) N(u). But N(c) and N(u) are integers and both c and u are in H , then both are larger than 1 and both divide p^2. The only way this is possible is that N(c) = N(u) =p.

Since $u \in H$, $u=m_0\zeta+m_1 i+m_2 j+m_3 k$ where $m_0, m_1, m_{2,} m_3$ $are\ integers$; thus $2u = 2\,m_0\zeta+2m_1 i+2m_2 j+2m_3 k = (m_0+m_0 i+m_0 j+m_0 k)+2m_1 i+2m_2 j+2m_3 k$ $= m_0+(2m_{1+}m_0)i+(2m_{2+}m_0)j+(2m_{3+}m_0)k$.

Therefore N(2u)=m_0^2+$(2m_{1+}m_0)^2$+$(2m_{2+}m_0)^2+(2m_{3+}m_0)^2$. But N(2u) = N(2)N(u) = 4p since N(2) = 4 and N(u) = p. We have shown that 4p= m_0^2+$(2m_{1+}m_0)^2$+$(2m_{2+}m_0)^2+(2m_{3+}m_0)^2$.

To complete the proof we have to use Euler's trick that if 2a=x_0^2 +x_1^2+$x_2^2+x_3^2$ where a, $x_0, x_1, x_{2,} x_3$ $are\ integers\ \ , then\ a = y_0^2$ +y_1^2+$y_2^2+y_3^2$ for some integers $y_0, y_1, y_{2,} y_3$. To see this note that as 2a is even all the x's are even or all are odd, or two are even and two are odd. At any rate in all three cases we can renumber the x's and pair them such a way that $y_0=\frac{x_0+x_1}{2}$, $y_1=\frac{x_0-x_1}{2}$, $y_2=\frac{x_2+x_3}{2}$, $y_3=\frac{x_2-x_3}{2}$ are all integers. But

$$y_0^2+y_1^2+y_2^2+y_3^2 = \left(\frac{x_0+x_1}{2}\right)^2+\left(\frac{x_0-x_1}{2}\right)^2+\left(\frac{x_2+x_3}{2}\right)^2+\left(\frac{x_2-x_3}{2}\right)^2$$

$$=\frac{1}{2}(x_0^2+x_1^2+x_2^2+x_3^2)=\frac{1}{2}(2a)=a.$$

Since 4p is a sum of four squares, then 2p is also so, As 2p is the sum of four squares so is p. Thus p=a_0^2 +a_1^2+$a_2^2+a_3^2$ for some integers $a_0, a_1, a_{2,} a_3$ so is the proof of the theorem.

Express an integer as sum of four squares:

Believe that Lagrange's four square theorem holds then how do you find the said integers . As earlier we give lemma III (**Lagrange's Identity**): If $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and $\beta_0, \beta_{1,} \beta_2, \beta_3$ are real numbers then

$(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2)(\beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2) = (\alpha_0\beta_0-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)^2+(\alpha_0\beta_1+\alpha_1\beta_0+\alpha_2\beta_3-\alpha_3\beta_2)^2+(\alpha_0\beta_2-\alpha_1\beta_3+\alpha_2\beta_0+\alpha_3\beta_1)^2+(\alpha_0\beta_3+\alpha_1\beta_2-\alpha_2\beta_1+\alpha_3\beta_0)^2$.

Here we put some examples as under:

11=$3^2+1^2+1^2+0^2$

2. 67=$8^2+1^2+1^2+1^2$

3. 751=$25^2+11^2+2^2+1^2$

4.

Let2176=2^7.17=2^6.34=64.34=$(4^2+4^2+4^2+4^2)(5^2+3^2+0^2+0^2)$=$(2012+0+0)^2(12+20+0-0)^2+(0-0+20+12)^2+(0+0-12+20)^2$ =$8^2+(32)^2+(32)^2+8^2$.

5. 1638=2.3^2.7.13=$(1^2+1^2+0^2+0^2)(1^2+1^2+1^2+0^2)^2(2^2+1^2+1^2+1^2)(3^2+2^2+0^2+0^2)$

= $\{(1^2+1^2+0^2+0^2)$ $(1^2+1^2+1^2+0^2)\}($ $1^2+1^2+1^2+0^2)$ $(2^2+1^2+1^2+1^2)(3^2+2^2+0^2+0^2)$

= $\{(0^2+2^2+1^2+1^2)\}$ $(1^2+1^2+1^2+0^2)$ $(2^2+1^2+1^2+1^2)$ $(3^2+2^2+0^2+0^2)$

= $\{(0^2+2^2+1^2+1^2)$ $(1^2+1^2+1^2+0^2)\}$ $(2^2+1^2+1^2+1^2)$ $(3^2+2^2+0^2+0^2)$

= $\{((-3)^2+1^2+2^2+2^2)\}$ $(2^2+1^2+1^2+1^2)$ $(3^2+2^2+0^2+0^2)$

= $\{((-3)^2+1^2+2^2+2^2)$ $(2^2+1^2+1^2+1^2)\}$ $(3^2+2^2+0^2+0^2)$

= $((-11)^2+(-1)^2+2^2+0^2)$ $(3^2+2^2+0^2+0^2)$

$= ((-31)^2 + (-25)^2 + 6^2 + 4^2$
$= 31^2 + 25^2 + 6^2 + 4^2$
Also1638=2.9.7.13=
$(1^2+1^2+0^2+0^2).(2^2+2^2+1^2+0^2).(2^2+1^2+1^2+1^2+).(3^2+2^2+0^2+0^2)$
$=\{(1^2+1^2+0^2+0^2).(2^2+2^2+1^2+0^2)\}.(2^2+1^2+1^2+1^2+).(3^2+2^2+0^2+0^2)$
$=(0^2+4^2+1^2+1^2)\ (2^2+1^2+1^2+1^2)(3^2+2^2+0^2+0^2)$
$=\{(0^2+4^2+1^2+1^2)(2^2+1^2+1^2+1^2)\}(3^2+2^2+0^2+0^2)=$ $((-6)^2+8^2+(-1)^2+5^2)(3^2+2^2+0^2+0^2)$
$= (-34)^2+12^2+7^2+17^2=34^2+12^2+7^2+17^2$

As above the integer 1638 has more than one representations i.e. in general there are many representations of a integer as a sum of 4 squares. Jacobi's four square theorem answer this problem.

Main Result : (Jacobi's Theorem)

The number of representations of an integer as the sum of four squares is equal to eight times the sum of all its divisors that are not divisible by 4. i.e. $r_{4(n)}=8\sum_{m:4|m|n} m$ or

The number of ways to represent an integer as the sum of four squares is eight times the sum of divisors of n if n is odd and 24 times the sum of the odd divisors of n if n is even.

i.e. $(r_{4(n)} = \begin{cases} 8\sum_{m/n} m & if\ n\ is\ odd \\ 24\sum_{m/n} m & if\ n\ is\ even \\ & m\ is\ odd \end{cases}$

Proof: Since the coefficient of x^n in the left side of the identity $(\sum_{n=-\infty}^{\infty} x^{n^2})^4$ $1+8\sum_{n=1}^{\infty}\frac{x^n}{[1+(-x)^n]^2}$, $|x|<1$ is $r_{4(n)}$ therefore

From the partition theory we have the following well known identities

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1})^2 = \sum_{n=-\infty}^{\infty} x^{n^2} \quad \text{......(1)}$$

$$2\prod_{n=1}^{\infty}\frac{(1-x^{2n})}{(1-x^{2n-1})} = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} \quad \text{......(2)}$$

$$\prod_{n=1}^{\infty}(1-x^n)^6 = \sum_{n=-\infty}^{\infty} x^{n^2}\sum_{n=0}^{\infty}(2n+1)^2 x^{n(n+1)} - \sum_{n=-\infty}^{\infty} x^{n(n+1)}\sum_{n=1}^{\infty}(2n)^2 x^{n^2} \quad \text{......(3)}$$

All are valid for all complex numbers x s.t. |x|<1 for proof of these one can see [1].

Now we realize that fourth power of right hand side for (1) also generate

Now from (3) we get $\prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{(1+x^{2n-1})^2}\sum_{n=0}^{\infty} r_{4(n)}x^n$ =

$$\prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{(1+x^{2n-1})^2}\prod_{n=1}^{\infty}(1-x^{2n})^4(1+x^{2n-1})^8 =$$
$$\sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}\sum_{n=0}^{\infty}(2n+1)^2 x^{n(n+1)} - \sum_{n=-\infty}^{\infty} x^{n(n+1)}\sum_{n=1}^{\infty}(-1)^n (2n)^2 x^{n^2}$$

Change $x \to x^4$ and multiplying by x we get

$$x\prod_{n=1}^{\infty}\frac{(1-x^{8n})^2}{(1+x^{8n-4})^2}\sum_{n=0}^{\infty} r_{4(n)}x^{4n} =$$
$$\sum_{n=-\infty}^{\infty}(-1)^n x^{4n^2}\sum_{n=0}^{\infty}(2n+1)^2 x^{(2n+1)^2} - \sum_{n=-\infty}^{\infty} x^{(2n+1)^2}\sum_{n=1}^{\infty}(-1)^n 4n^2 x^{4n^2} \quad \text{......(4)}$$

Define G(x) as s.t. $2G(x) = \frac{\sum_{n=-\infty}^{\infty} x^{(2n+1)^2}}{\sum_{n=-\infty}^{\infty}(-1)^n x^{4n^2}}$

$$= \frac{2x\prod_{n=1}^{\infty}\frac{1-x^{16n}}{1-x^{16n-8}}}{\prod_{n=1}^{\infty}(1-x^{8n})(1-x^{8n-4})^2} =$$

$$2x\prod_{n=1}^{\infty}(1-x^{16n-8})^{-2}(1-x^{8n-4})^{-2} \quad \text{.................(*)}$$

[By using (1) and(2)]Differentiating (*) w.r.t. x and multiplying by x we get

xG/(x)= 〖{∑_(n = −∞)^∞▒〖(−1)〗^n 〖x^4n〗^2 }〗^(−2) {∑_(n = 0)^∞▒〖〖(2n + 1)〗^2 x^(〖(2n + 1)〗^2)〗 ∑_(n = −∞)^∞▒〖(−1)〗^n 〖x^4n〗^2 − ∑_(n = −∞)^∞▒x^(〖(2n + 1)〗^2) ∑_(n = 1)^∞▒〖(−1)〗^n 〖〖〖4n〗^2 x〗^4n〗^2 }

Therefore R.H.S. of (4) becomes,

$$x\left\{\sum_{n=-\infty}^{\infty}(-1)^n x^{4n^2}\right\}^{-2} G^{/}(x) = x\prod_{n=1}^{\infty}(1-x^{8n})^2(1-x^{8n-4})^4$$

$G^{/}(x)$

Put into (4) we get, $G^{/}(x) = \prod_{n=1}^{\infty}(1+x^{8n-4})^{-2}(1-x^{8n-4})^{-4}\sum_{n=0}^{\infty} r_4^{(n)}x^{4n}$

$=\prod_{n=1}^{\infty}(1-x^{16n-8})^{-2}(1-x^{8n-4})^{-2}\sum_{n=0}^{\infty} r_4^{(n)}x^{4n} =$

$x^{-1}G(x)\sum_{n=0}^{\infty} r_4^{(n)}x^{4n}$ ....(5)

Using logarithmic differentiation we get

$x\frac{d}{dx}\{\log G(x)\} = x\,\frac{G^{/}(x)}{G(X)} = 1+16\sum_{n=1}^{\infty}\frac{(2n-1)x^{8(2n-1)}}{1-x^{8(2n-1)}} + 8\sum_{n=1}^{\infty}\frac{(2n-1)x^{4(2n-1)}}{1-x^{4(2n-1)}}$

$=1+16\sum_{n=1}^{\infty} x^{8n}\sum_{d/n} d$,d is odd $+ 8\sum_{n=1}^{\infty} x^{4n}\sum_{d/n} d$,d is odd

$=1+16\sum_{n=1}^{\infty}\sigma(o(n))x^{8n} + 8\sum_{n=1}^{\infty}\sigma(o(n))x^{4n} = 1+\sum_{n=1}^{\infty}\{16\sigma(o(n)) + 8\sigma(o(2n))\}x^{8n} +$

$8\sum_{n=1}^{\infty}\sigma(2n-1)x^{8n-4}$

=1+ $24\sum_{n=1}^{\infty}\sigma\big(o(2n)\big)x^{8n}$+ $8 \sum_{n=1}^{\infty}\sigma(2n-1)x^{8n-1}$

Thus$G^{/(}$x)

= x^{-1} G(x) $\{1+24\sum_{n=1}^{\infty}\sigma\big(o(2n)\big)x^{8n}+8\sum_{n=1}^{\infty}\sigma\big(o(2n-1)\big)x^{8n-4}\}$ ....(6)

Eliminating $G^{/}(x)$ from (5) and (6) we get

x^{-1} G(x) $\sum_{n=0}^{\infty} r_4^{(n)}\, x^{4n}$=$x^{-1}$G(x) $\{1+24\sum_{n=1}^{\infty}\sigma\big(o(2n)\big)x^{8n}+8\sum_{n=1}^{\infty}\sigma\big(o(2n-1)\big)x^{8n-4}\}$

We get

$\sum_{n=0}^{\infty} r_4^{(n)}\, x^{4n}$=$1+24\sum_{n=1}^{\infty}\sigma\big(o(2n)\big)x^{8n}+8\sum_{n=1}^{\infty}\sigma\big(o(2n-1)\big)x^{8n-4}$

Equating the coefficients of like powers we get

$r_4^{(2n-1)}$=8σ(2n-1) and $r_4^{(2n)}$ =24σ(o(2n))

Or $r_4^{(n)}$=

$$\begin{cases} 8\sum_{m/n} m \text{ , if n is odd} \\ 24\sum_{m/n} m \text{ , if n is even and m is odd} \end{cases}$$

Or

$r_4^{(n)}$=8$\sum d$ where d/n and $4 \nmid d$. This completes the proof of the theorem.

Application: For the above said integer 1638 its divisors the combinations of 2$\times 3^2 \times 7 \times 13$ out of which that are not divisible by 4 are 1,2,3,6,7,9,13,14,18,21,26,39,42,63,78,91,117,126,182,234, 273,546,819,1638 whose sum is 4368 ,therefore total number of representations of 1638 as the sum of squares of four integers are 8$\times$ 4368 = 34944

Similarly for the integer 2176 =$2^7 \times 17$ the divisors of which are not divisible by 4 are 1,2,17,34

Therefore numbers of representations are 8$\times (1+2+17+34) = 8(54) = 432$

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