COEFFICIENT INEQUALITY FOR A NEW CLASS OF ANALYTIC FUNCTIONS

GURMEET SINGH, JYOTI GARG

Abstract: We Interoduce a new class of analytic functions and obtain Fekete-Szego Inequality for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes.We also find extremal functions which make the inequality sharp.

Keywords: Univalent functions, Analytic functions,bounded functions and FEKETE-SZEGO Inequality.

1. Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \quad (1.4)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.5)$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

We introduce a new class as

$$\left\{(f(z) \in \mathcal{A}; \frac{zf'(f(z))f'(z)}{f(f(z))} \prec \frac{1+z}{1-z}, z \in \mathbb{E}\right\}$$

(1.6)

And we will denote this class as $S^*(f(f(z)))$.

We will establish coefficient inequality for this class.The two important subclasses of $S^*(f(f(z))$ are

$$S^*(f(f(z),\delta) = \left\{f(z) \in \mathcal{A}; \frac{zf'(f(z))f'(z)}{f(f(z))} \prec (\frac{1+z}{1-z})^{\delta}, z \in \mathbb{E}\right\} \quad (1.7)$$

$$S^*(f(f(z)), A, B) = \left\{f(z) \in \mathcal{A}; \frac{Zf'(f(z))f'(z)}{f(f(z))} \prec \frac{1+Az}{1+bz}, z \in \mathbb{E}\right\}$$

(1.8)

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z)); z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1$.

(1.9)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$. (1.10)

2. PRELIMINARY LEMMAS:

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$\frac{1+w(z)}{1+w(z)} = 1 + 2cz + 2z^2 + - - -$ (2.1)

3. Main Results

THEOREM 3.1: Let $f(z) \in S^*(f(f(z))$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \delta\left(\frac{1}{2} - \delta\mu\right) & \mu \leq 0 \\ \frac{1}{2}\delta & 0 \leq \mu \leq 1 \\ \delta\left(\frac{1}{2} + \delta(\mu - 1)\right) & \mu \geq 1 \end{cases}$$

Proof: By definition of $f(z) \in S^*(f(f(z))$,

$\frac{zf'(f(z))f'(z)}{f(f(z))} = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta}$ (3.2)

we have expanding the series (3.4), we get

$1 + 4a_2 z + (6a_3 + 6a_2^2)z^2 + - - - - - - - - - - = \left(1 + (2a_2 + 2\delta c_1)z + 2\delta^2 c_1^2 + 2a_3 + 2a_2^2 + 4\delta a_2 c_1 + 2\delta c_2 - - - \right).$ (3.3)

Identifying terms in (3.5), we get

$a_2 = \delta c_1$

(3.4)

$a_3 = \frac{\delta^2 c_1^2 + \delta c_2}{2}$

(3.5)

From (3.6) and (3.7), we obtain

$a_3 - \mu a_2^2 = \frac{\delta}{2}c_2 + \left(\frac{\delta^2}{2} - \mu\delta^2\right)c_1^2$

(3.6)

Taking absolute value, (3.8) can be rewritten as

$$\left|a_3 - \mu a_2^2\right| \leq \left|\frac{1}{2}\delta\right| |c_2| + \left|\frac{\delta^2}{2} - \mu\delta^2\right| |c_1|^2$$

(3.7)

Using (1.10) in (3.7), we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{2}\delta(1 - |c_1|^2 + \left|\frac{\delta^2}{2} - \mu\delta^2\right| |c_1^2|$$

$= \frac{1}{2}\delta + \left[\left|\frac{\delta^2}{2} - \mu\delta^2\right| - \frac{1}{2}\delta\right]|c_1|^2.$ (3.8)

Case I: $\mu \leq \frac{1}{2}$ (3.8) can be rewritten as

$\frac{1}{\delta}\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2} - \delta\mu|c_1|^2.$ (3.9)

Subcase I (a): $\mu \leq 0$ Using (1.10), (3.11) becomes

$\frac{1}{\delta}\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2} - \delta\mu(1)$ (3.10)

Subcase I (b): $\mu \geq 0$ We obtain from (3.10)

$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2}\delta$ (3.11)

Case II: $\mu \geq \frac{1}{2}$

Preceding as in case I, we get

$\frac{1}{\delta}\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2} + \delta(\mu - 1)|c_1|^2.$ (3.12)

Subcase II (a): $\mu \leq 1$

(3.12) takes the form $\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2}\delta$ (3.13)

Subcase II (b): $\mu \geq 1$

Preceding as in subcase I (a), we get

$\left|a_3 - \mu a_2^2\right| \leq \delta\left[\frac{1}{2} + \delta(\mu - 1\right].$ (3.14)

Thus the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = z\left(1 + z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + - - -\right)$$

Extremal function for (3.2) is defined by $f_2(z) = \frac{z}{\sqrt{1-z^2}}$

Corollary 3.2: Putting $\delta = 1$ in the theorem, we get

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{2} - \mu, if \mu \leq 0; \\ \frac{1}{2}, if 0 \leq \mu \leq 1; \\ \mu - \frac{1}{2}, if\ \mu \geq 1 \end{cases}$$

These are the results of $S^*(f(f(z))$

References:

1. Chichra, P. N., "New subclasses of the class of close-to-convex functions", Procedure of American Mathematical Society, 62, (1977), 37-43.
2. Goel, R. M. and Mehrok, B. S., "A subclass of univalent functions", Houston Journal of Mathematics, 8, (1982), 343-357.
3. Kaplan, W., "Close-to-convex schlicht functions", Michigan Mathematical Journal, 1, (1952), 169-185.
4. Keogh, S. R. and Merkes, E. R., "A Coefficient inequality for certain classes of analytic functions", Procedure of American Mathematical Society, 20, (1989), 8-12.
5. K. Löwner, "Uber monotone Matrixfunktionen", Math. Z., 38, (1934), 177-216.
6. Kunle Oladeji Babalola, "The fifth and sixth coefficients of α-close-to-convex functions", Kragujevac J. Math., 32, (2009), 5-12.

7. L. Bieberbach, “Uber einige Extremal probleme im Gebiete der konformen Abbildung”, Math. Ann., 77(1916), 153-172.
8. L. Bieberbach, “Uber die Koeffizientem derjenigen Potenzreihen, welche eine schlithe Abbildung des Einheitskreises vermitteln”, Preuss. Akad. Wiss. Sitzungsb., (1916), 940-955.
9. M. Fekete and G. Szegö, “Eine Bemerkung über ungerade schlichte Funktionen”, J. London Math. Soc. 8 (1933), 85-89.
10. R. M. Goel and B. S.Mehrok, “A coefficient inequality for a subclass of closeto-convex functions”, Serdica Bul. Math. Pubs., 15, (1989), 327-335.
11. Z. Nehari, “Conformal Mapping”, McGraw-Hill, Comp., Inc., New York, (1952).

KHALSA COLLEGE, PATIALA
(e-mail: meetgur111@gmail.com)
JYOTI GARG
KHALSA COLLEGE, PATIALA
(e-mail: jyoti90.garg2@gmail.com)