FEKETE-SZEGÖ INEQUALITY OF A NEW SUBCLASS OF CONVEX AND STARLIKE FUNCTIONS

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Abstract: In this paper, we will introduce a new subclass of convex and starlike functions and will obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to this subclass.

Keywords: Univalent functions, Starlike functions, convex functions and bounded functions.

Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Definitions:

Convex Function: A set $E \subseteq \mathbb{C}$ is said to be convex if the line segment joining any two points of E lies entirely in E.

A function is said to be convex if it maps the unit disc Δ onto a convex domain.

Starlike Function: A set E is said to be starlike w.r.t. point α ∈ E if the line segment joining α to every other point $w \in E$ lies entirely in E.

A function is said to be starlike w.r.t. α if it maps the unit disc Δ onto a domain starlike w.r.t. α .

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

$\in \mathcal{A}$ and satisfying the condition

$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.3)

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

$\in \mathcal{A}$ and satisfying the condition

$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}.$ (1.4)

Introducing a new subclass of convex and starlike functions, denoted by $\mathcal{CSK}^*$, as $\left\{\mathbf{f(z)} \in \mathcal{A}; \frac{\mathbf{z\{\alpha f'(z)+(1-\alpha)(zf'(z))'\}}}{\mathbf{\alpha f(z)+(1-\alpha)zf'(z)}} \prec \frac{\mathbf{1+z}}{\mathbf{1-z}}; \mathbf{z} \in \mathbb{E}\right\}$.

It should be noted that

$\mathcal{CSK}^*(\mathbf{0}) = \mathcal{K}$ **and** $\mathcal{CSK}^*(\mathbf{1}) = \mathcal{S}^*$

The symbol $\prec$ stands for subordination, which is defined below:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.5)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$. (1.6)

2. PRELIMINARY LEMMAS:

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$\frac{1+w(z)}{1-w(z)} = 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + --$

Here $|c_1| \leq 1, |c_2| \leq$ 1- $|c_1|^2$ (2.1)

3. MAIN RESULTS

THEOREM 3.1: If $\boldsymbol{f(z)} \in \mathcal{CSK}^*$, then

$\left|a_3-\mu a_2^2\right| \leq$

$$\begin{cases} \frac{3}{3-2\alpha}-\frac{4\mu}{(2-\alpha)^2}\ ,\ if\ \mu \leq \frac{(2-\alpha)^2}{2(3-2\alpha)};\ (3.1) \\ \frac{1}{3-2\alpha}\ ,\quad if\ \frac{(2-\alpha)^2}{2(3-2\alpha)} \leq \mu \leq \frac{(2-\alpha)^2}{(3-2\alpha)};\ (3.2) \\ \frac{-3}{3-2\alpha}+\frac{4\mu}{(2-\alpha)^2}\ ,\ if\ \mu \geq \frac{(2-\alpha)^2}{(3-2\alpha)}.(3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $\mathcal{CSK}^*$, we have

$$\frac{z\{\alpha f'(z)+(1-\alpha)(zf'(z))'\}}{\alpha f(z)+(1-\alpha)zf'(z)}=\frac{1+w(z)}{1-w(z)};w(z)\in\mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we have

$$\alpha\{1+2a_2z+3a_3z^2+\cdots\}+(1-\alpha)\{1+4a_2z+9a_3z^2+\cdots\}=\{\alpha(1+a_2z+a_3z^2+\cdots)+(1-\alpha)(1+2a_2z+3a_3z^2+\cdots)\}\{1+2c_1z+2(c_2+c_1^2)z^2+\cdots\} \quad (3.5)$$

Comparing coefficient of z for a_2 and coefficient of z^2 for a_3 , to get

$$a_2=\frac{2c_1}{(2-\alpha)} \quad (3.6)$$

$$a_3=\frac{(c_2+3c_1^2)}{(3-2\alpha)} \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_{3-}\mu a_2^2=\frac{c_2}{(3-2\alpha)}+\left(\frac{3}{(3-2\alpha)}-\frac{4\mu}{(2-\alpha)^2}\right)c_1^2 \quad (3.8)$$

Using the fact $|c_1|\leq 1, |c_2| \leq$ 1- $|c_1|^2$,

We get

$$\left|a_{3-}\mu a_2^2\right| \leq \frac{1}{(3-2\alpha)}+\left(\left|\frac{3}{(3-2\alpha)}-\frac{4\mu}{(2-\alpha)^2}\right|-\frac{1}{(3-2\alpha)}\right)\left|c_1^2\right| \quad (3.9)$$

Case I: If $\frac{3}{(3-2\alpha)}-\frac{4\mu}{(2-\alpha)^2}\geq 0$,

Then

$$\left|a_{3-}\mu a_2^2\right| \leq \frac{1}{(3-2\alpha)}+\left(\frac{2}{(3-2\alpha)}-\frac{4\mu}{(2-\alpha)^2}\right)\left|c_1^2\right| \quad (3.10)$$

Subcase I (a): If $\mu \leq \frac{(2-\alpha)^2}{2(3-2\alpha)}$,

Then $\left(\frac{2}{(3-2\alpha)}-\frac{4\mu}{(2-\alpha)^2}\right)\geq 0.$

Since $\left|c_1^2\right|\leq 1$, (3.10) becomes

$$\left|a_{3-}\mu a_2^2\right| \leq \frac{3}{(3-2\alpha)}-\frac{4\mu}{(2-\alpha)^2} \quad (3.11)$$

Subcase I (b): If $\mu \geq \frac{(2-\alpha)^2}{2(3-2\alpha)}$,

Then $\left(\frac{2}{(3-2\alpha)}-\frac{4\mu}{(2-\alpha)^2}\right)\leq 0.$

Simply ignoring this negative term, the equation (3.10) becomes

$$\left|a_3-\mu a_2^2\right| \leq \frac{1}{3-2\alpha} \quad (3.12)$$

Case II: If $\frac{3}{(3-2\alpha)}-\frac{4\mu}{(2-\alpha)^2}\leq 0$,

Then

$$\left|a_{3-}\mu a_2^2\right| \leq \frac{1}{(3-2\alpha)}+\left(\frac{-4}{(3-2\alpha)}+\frac{4\mu}{(2-\alpha)^2}\right)\left|c_1^2\right| \quad (3.13)$$

Subcase II (a): If $\mu \geq \frac{(2-\alpha)^2}{(3-2\alpha)}$,

then $\left(\frac{-4}{(3-2\alpha)}+\frac{4\mu}{(2-\alpha)^2}\right)\geq 0$.

Since $\left|c_1^2\right|\leq 1$, (3.13) becomes

$$\left|a_{3-}\mu a_2^2\right| \leq \frac{-3}{(3-2\alpha)}+\frac{4\mu}{(2-\alpha)^2} \quad (3.14)$$

Subcase II (b): If $\mu \leq \frac{(2-\alpha)^2}{(3-2\alpha)}$,

Then $\left(\frac{-4}{(3-2\alpha)}+\frac{4\mu}{(2-\alpha)^2}\right)\leq 0.$

Simply ignoring this negative term, the equation (3.13) becomes

$$\left|a_3-\mu a_2^2\right| \leq \frac{1}{3-2\alpha} \quad (3.15)$$

This completes the theorem. The results are sharp.

Extremal function for (3.1) and (3.3) is given by

$$f_1(z)=z\left(1-\frac{(3\alpha^2-8\alpha+6)}{(2-\alpha)(3-2\alpha)}z\right)^{\frac{2(2\alpha-3)}{3\alpha^2-8\alpha+6}}.$$

Extremal function for (3.2) is given by

$f_2(z)=z(1+z)^{1/(3-2\alpha)}$.

Corollary 3.2: Putting $\alpha=0$ in the theorem, we get

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 1-\mu\ , & if\ \mu \leq \frac{2}{3}; \\ \frac{1}{3}\ , & if\ \frac{2}{3}\leq \mu \leq \frac{4}{3}; \\ -1+\mu\ , & if\ \mu \geq \frac{4}{3}. \end{cases}$$

Corollary 3.3: Putting $\alpha=1$ in the theorem, we get

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 3-4\mu\ , & if\ \mu \leq \frac{1}{2}; \\ 1\ , & if\ \frac{1}{2}\leq \mu \leq 1; \\ -3+4\mu\ , & if\ \mu \geq 1. \end{cases}$$

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