# FEKETE-SZEGÖ INEQUALITY FOR A NEW CLASS OF REGULAR FUNCTION

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**Abstract:** We will describe a subclass of p-valent analytic functions in this paper and will obtain sharp upper bounds of the functional $|a_{p+2}-\mu a_{p+1}^2|$ for the analytic function $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, |z| < 1$ belonging to this subclass.

**Keywords:** Univalent functions, Starlike functions, Close to convex functions and bounded functions.

**Introduction :** Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between $a_3$ and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3-4\mu, if\ \mu \leq 0; \\ 1+2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \qquad (1.4)$$

**p-VALENT FUNCTION**: Multivalent functions and in particular p-valent functions, are a generalization of univalent functions. In the study of univalent functions, one of the fundamental problems is whether there exists a univalent mapping from a given domain $E$ onto a given domain $D$. A necessary condition for the existence of such a mapping is that $E$ and $D$ have equal degrees of connectivity. If $E$ and $D$ are simply-connected domains whose boundaries contain more than one point, then this condition is also sufficient and the problem reduces to mapping a given domain onto a disc. In this connection, a special role is played in the theory of univalent functions on simply-connected domains by the $S$, class of functions $f$ that are regular and univalent on the unit disc $E = \{z: |z| < 1\}$, normalized by the conditions $f(0) = 0, f'(0) = 1$, and having the expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + - - -, z \in E$$

In the case of multiply-connected domains, mappings of a given multiply-connected domain onto so-called canonical domains are studied. In particular, p-valent functions can be defined as follow:

Let $\mathcal{A}_p$ (p is a positive integer) denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$$

which are analytic in the unit disc $E$. Clearly, $\mathcal{A}_1 = \mathcal{A}$. A function $f(z) \in \mathcal{A}_p$ is said to be p-valent in $E$ if it assumes no value more than p times in $E$.

**p-VALENT STARLIKE FUNCTION**:

A function $f(z) \in \mathcal{A}_p$ is said to be a p-valent starlike function in $E$ if there exists a positive real number ρ such that

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0$$

and

$$\int_0^{\pi}\left[Re\left(\frac{zf'(z)}{f(z)}\right)\right]d\theta = 2p\pi, z = re^{i\theta} for$$

$$\rho < |z| < 1.$$

We denote the class of p-valent starlike functions by $S_p^*$. By $S_p^*(\beta)$, we denote the class of functions $f(z) \in \boldsymbol{\mathcal{A}_p}$ satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta; 0 \le \beta < p, z \in E$$

**Note:** p-valent starlike functions are also called p-valently starlike functions.

$f(z) \in S_p^*(\beta)$ is called p-valently starlike function of order $\beta$.

We introduce a new subclass as $\left\{\mathbf{f(z)} \in \boldsymbol{\mathcal{A}_p}; \frac{\left[\mathbf{z\{zf'(z)\}'}\right]'}{\mathbf{p\{zf'(z)\}'}} \prec \frac{\mathbf{1+Az}}{\mathbf{1+Bz}}; \mathbf{z} \in \mathbb{E}\right\}$ and we will denote this class as $f(z) \in \mathcal{H}_p^*$.

Symbol $\prec$ stands for subordination, which we define as follows:

**Principle of Subordination:** Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1$. (1.5)

It is known that $|d_1| \le 1, |d_2| \le 1 - |d_1|^2$. (1.6)

## 2. PRELIMINARY LEMMAS:

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + --- \quad (2.1)$$

Here $|c_1| \le 1, |c_2| \le 1 - |c_1|^2$ (2.2)

## 3. MAIN RESULTS

**THEOREM 3.1**: Let $f(z) \in \mathcal{H}_p^*$, then

$$|a_{p+2} - \mu a_{p+1}^2| \le \begin{cases} \frac{(A-B)p^3[(A-B)p-B]}{2(p+2)^2} - \mu\frac{(A-B)^2p^6}{(p+1)^4} \\ if\ \mu \le \frac{[(A-B)p-(B+1)](p+1)^4}{2(p+2)^2(A-B)p^3} & (3.1) \\ \frac{(A-B)p^3}{2(p+2)^2} \\ if\ \frac{[(A-B)p-(B+1)](p+1)^4}{2(p+2)^2(A-B)p^3} \le \mu \le \frac{[1-B+(A-B)p](p+1)^4}{2(p+2)^2(A-B)p^3} & (3.2) \\ \mu\frac{(A-B)^2p^6}{(p+1)^4} - \frac{(A-B)p^3[(A-B)p-B]}{2(p+2)^2} \\ if\ \mu \ge \frac{[1-B+(A-B)p](p+1)^4}{2(p+2)^2(A-B)p^3} & (3.3) \end{cases}$$

The results are sharp.

**Proof:** By definition of $f(z) \in \mathcal{H}_p^*$, we have

$$\frac{\left[\mathbf{z\{zf'(z)\}'}\right]'}{\mathbf{p\{zf'(z)\}'}} = \frac{1+Az}{1+Bz}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$p^3z^{p-1} + a_{p+1}(p+1)^3z^p + a_{p+2}(p+2)^3z^{p+1} + --- = (1 + c_1(A-B)z + (A-B)(c_2-Bc_1^2)z^2 + ---)(p^3z^{p-1} + a_{p+1}(p+1)^2p\,z^p + a_{p+2}(p+2)^2p\,z^{p+1} + ---) \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_{p+1} = \frac{c_1(A-B)p^3}{(p+1)^2} \quad (3.6)$$

$$a_{p+2} = \frac{{c_1}^2(A-B)^2p^4 + (A-B)(c_2-Bc_1^2)p^3}{2(p+2)^2} \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{{c_1}^2(A-B)^2p^4 + (A-B)(c_2-Bc_1^2)p^3}{2(p+2)^2} - \mu\frac{{c_1}^2(A-B)^2p^6}{(p+1)^4} \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{|c_2|(A-B)p^3}{2(p+2)^2} + \left|\frac{(A-B)^2p^4 - B(A-B)p^3}{2(p+2)^2} - \mu\frac{(A-B)^2p^6}{(p+1)^4}\right||c_1|^2$$

(3.9)

Using (2.2) in (3.9), we get

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{(1-|c_1|^2)(A-B)p^3}{2(p+2)^2} + \left|\frac{(A-B)^2p^4 - B(A-B)p^3}{2(p+2)^2} - \mu\frac{(A-B)^2p^6}{(p+1)^4}\right||c_1|^2$$

(3.10)

**Case I**: $\mu \le \frac{[(A-B)p-B](p+1)^4}{2(p+2)^2(A-B)p^3}$

(3.10) can be rewritten as

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{(1-|c_1|^2)(A-B)p^3}{2(p+2)^2} + \left(\frac{(A-B)^2p^4 - B(A-B)p^3}{2(p+2)^2} - \mu\frac{(A-B)^2p^6}{(p+1)^4}\right)|c_1|^2$$

$$|a_{p+2} - \mu a_{p+1}^2| \le$$

$\frac{(A-B)p^3}{2(p+2)^2}$ + $\left(\frac{(A-B)^2p^4-(B+1)(A-B)p^3}{2(p+2)^2} - \mu\ \frac{(A-B)^2p^6}{(p+1)^4}\right)|c_1|^2$ (3.11)

**Subcase I (a)**: $\mu \le \frac{[(A-B)p-(B+1)](p+1)^4}{2(p+2)^2(A-B)p^3}$

Using (2.2), (3.11) becomes

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{(A-B)p^3[(A-B)p-B]}{2(p+2)^2} - \mu\frac{(A-B)^2p^6}{(p+1)^4} \quad (3.12)$$

**Subcase I (b)**: $\mu \ge \frac{[(A-B)p-(B+1)](p+1)^4}{2(p+2)^2(A-B)p^3}$. We obtain from (3.11)

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{(A-B)p^3}{2(p+2)^2} \quad (3.13)$$

**Case II**: : $\mu \ge \frac{[(A-B)p-B](p+1)^4}{2(p+2)^2(A-B)p^3}$

Preceding as in case I, we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{(A-B)p^3}{2(p+2)^2} - \left(\frac{(A-B)^2p^4+(1-B)(A-B)p^3}{2(p+2)^2} - \mu\ \frac{(A-B)^2p^6}{(p+1)^4}\right)|c_1|^2 \quad (3.14)$$

**Subcase II (a)**: $\mu \ge \frac{[1-B+(A-B)p](p+1)^4}{2(p+2)^2(A-B)p^3}$

(3.14) takes the form $|a_{p+2} - \mu a_{p+1}^2| \le$

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{(A-B)p^3}{2(p+2)^2} - \left(\frac{(A-B)^2p^4+(1-B)(A-B)p^3}{2(p+2)^2} - \mu\ \frac{(A-B)^2p^6}{(p+1)^4}\right)$$

$$|a_{p+2} - \mu a_{p+1}^2| \le \mu\frac{(A-B)^2p^6}{(p+1)^4} - \frac{(A-B)p^3[(A-B)p-B]}{2(p+2)^2} \quad (3.15)$$

**Subcase II (b)**: $\mu \le \frac{[1-B+(A-B)p](p+1)^4}{2(p+2)^2(A-B)p^3}$

Preceding as in subcase I (b), we get

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{(A-B)p^3}{2(p+2)^2} \quad (3.16)$$

Combining (3.12),(3.13), (3.15) and (3.16), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by $f_1(z) = (1+az)^n$ where $a = \frac{(A-B)p^3(p+2)^2-(p+1)^4((A-B)p-B)}{(p+1)^2(p+2)^2}$

And n= $\frac{(A-B)p^3(p+2)^2}{(A-B)p^3(p+2)^2-(p+1)^4((A-B)p-B)}$

Extremal function for (3.2) is defined by

$$f_2(z) = z\left(1 + \frac{p^3z}{2(p+2)^2}\right)^{(A-B)}$$

**Corollary 3.2:** Putting $A = 1, B = -1$ in the theorem, we get

$$\left|a_{p+2-\mu a_{p+1}^2}\right| \le \begin{cases} \frac{p^3(\ 2p+1)}{(p+2)^2} - \frac{4up^6}{(p+1)^4}\ if\mu \le \frac{(\ p+1\ )^4}{2p^2(p+2)^2} \\ \frac{p^3}{(p+2)^2}\ if\ \frac{(\ p+1\ )^4}{2p^2(p+2)^2} \le \mu \le \frac{(p+1)^5}{2p^3(p+2)^2} \\ \frac{4up^6}{(p+1)^4} - \frac{p^3(\ 2p+1)}{(p+2)^2}\ if\ \mu \ge \frac{(p+1)^5}{2p^3(p+2)^2} \end{cases}$$

These estimates were derived by Rathore and Ravinder [3]

**Corollary 3.3:** Putting $p = 1$ in the theorem, we get

$$|a_{3-}\mu a_2{}^2| \le \begin{cases} \frac{(A-B)(A-2B)}{18} - \mu\frac{(A-B)^2}{16}\ if\ \mu \le \frac{8(A-2B-1)}{9(A-B)} \\ \frac{(A-B)}{18}\ if\ \frac{8(A-2B-1)}{9(A-B)} \le \mu \le \frac{8(A-2B+1)}{9(A-B)} \\ \frac{\mu(A-B)^2}{16} - \frac{(A-B)(A-2B)}{18}\ if\mu \ge \frac{8(A-2B+1)}{9(A-B)} \end{cases}$$

**Corollary 3.4:** Putting $A = 1, B = -1, p = 1$ in the theorem, we get

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{3} - \frac{\mu}{4}\ if\ \ \mu \le \frac{8}{9} \\ \frac{1}{9}\ if\ \ \ \frac{8}{9} \le \mu \le \frac{16}{9} \\ -\frac{1}{3} + \frac{\mu}{4}\ if\ \mu \ge \frac{16}{9} \end{cases}$$

These estimates were derived by Rathore and Ravinder [3]

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