FEKETE-SZEGÖ INEQUALITY FOR CERTAIN SUBCLASS OF STARLIKE AND INVERSE STARLIKE ANALYTIC FUNCTIONS

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Abstract: Here we describe some classes of analytic functions and its subclasses by which we will be obtaining sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

1. Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases} \quad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$ and satisfying the condition

$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.3)

We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$ and satisfying the condition

$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}.$ (1.4)

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$Re\left(\frac{zf\prime(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.5)

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$ (1.6)

$\mathcal{K}(A,B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))\prime}{f\prime(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$ (1.7)

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclasses $\left\{\mathbf{f(z)} \in \mathcal{A}; \boldsymbol{\alpha}\frac{\mathbf{zf'(z)}}{\mathbf{f(z)}} + (\mathbf{1} - \boldsymbol{\alpha})\frac{\mathbf{zf(z)}}{\mathbf{2}\int_0^z \mathbf{f(z)dz}} \prec \frac{\mathbf{1+Az}}{\mathbf{1+Bz}}; \mathbf{z} \in \mathbb{E}\right\}$ and we will denote this class as $f(z) \in \mathcal{C}(S^*)^{-1}[A,B]$,

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.8)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2.$ (1.9)

2. PRELIMINARY LEMMAS: For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that $\frac{1+Aw(z)}{1+Bw(z)} = 1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2)z^2 + - - - (2.1)$

3. MAIN RESULT:

THEOREM 3.1: Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S^*(f, f', \alpha, \beta, \delta)$., then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta}{3\alpha+\beta-4\alpha\beta} + f[g - 4\mu\delta]\delta, if \mu \leq w; \\ \frac{\delta}{3\alpha+\beta-4\alpha\beta} \quad if w \leq \mu \leq y; \\ \frac{\delta}{3\alpha+\beta-4\alpha\beta} + f[4\mu\delta - h]\delta, if \mu \geq y \end{cases}$$

Where $f = \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}$

$w = \frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta-\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4(3\alpha+\beta-4\alpha\beta)\delta}$,

$y = \frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta+\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4(3\alpha+\beta-4\alpha\beta)\delta}$, $g = \frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta-\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}$ $h = \frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta+\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}$.

The result is sharp in the sense that right hand side is the least upper bound of the result and extremal function exists.

PROOF: By definition of $S^*(f, f', \alpha, \beta, \delta)$, we have

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \alpha\left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta}; w(z) \in \mathcal{U}. \quad (3.1)$$

Expanding the series (3.1), we get

$$(1-\alpha)\left\{1 + \beta a_2 z + (2\beta a_3 + \frac{\beta(\beta-3)}{2} a_{12}^2)z^2 + - - -\right\} + \alpha\{1 + 2(1-\beta)a_2 z + 2(1-\beta)(3a_3 - (\beta+2)a_2^2)z^2 + - - -\} = (1 + 2\delta c_1 z + 2\delta(c_2 + \delta {c_1}^2)z^2 + - - -). \quad (3.2)$$

Comparing the coefficients of like powers in equation (3.2), we get

$a_2 = \frac{2\delta}{(1-\alpha)\beta+2\alpha(1-\beta)} c_1$ and

$$a_3 = \frac{\delta}{3\alpha+\beta-4\alpha\beta} c_2 + \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2}{(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} {c_1}^2 \delta^2. \quad (3.3)$$

From equation (3.3), we obtain

$$a_3 - \mu a_2^2 = \frac{\delta}{3\alpha+\beta-4\alpha\beta} c_2 + \left[\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2}{(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} - \frac{4}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\mu\right]\delta^2 c_1^2.$$

(3.4)

Taking absolute value, it can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{\delta}{3\alpha+\beta-4\alpha\beta}|c_2| + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right|\delta^2|c_1^2|.$$

(3.5)

Using (1.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{\delta}{3\alpha+\beta-4\alpha\beta}(1-|c_1|^2) + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right||c_1^2|\delta^2.$$

$$= \frac{\delta}{3\alpha+\beta-4\alpha\beta} + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right|\delta^2 - \frac{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2\delta}{3\alpha+\beta-4\alpha\beta}\right]|c_1|^2.$$

Case I:

$\mu \leq \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2}{4(3\alpha+\beta-4\alpha\beta)}$.

Above inequality can be rewritten as $\left|a_3-\mu a_2^2\right| \leq \frac{\delta}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta-\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}-4\mu\delta\right]\delta|c_1|^2$(3.6)

SubcaseI(a):$\mu \leq w$, where $w=\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta-\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4(3\alpha+\beta-4\alpha\beta)\delta}$. Using (1.9), we get from inequality (3.6) that$\left|a_3-\mu a_2^2\right| \leq$
$\frac{\delta}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta-\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}-4\mu\delta\right]\delta$(3.7)

Subcase I (b):$\mu \geq w$,
where $w=\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta-\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4(3\alpha+\beta-4\alpha\beta)\delta}$. We obtain from inequality (3.6) that
$\left|a_3-\mu a_2^2\right| \leq \frac{\delta}{3\alpha+\beta-4\alpha\beta}$. (3.8)

CaseII:$\mu \geq \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2}{4(3\alpha+\beta-4\alpha\beta)}$

Preceding as in case I, we get

$$\left|a_3-\mu a_2^2\right| \leq \frac{\delta}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[4\mu\delta-\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta+\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}\right]\delta|c_1|^2$$

(3.9)

Subcase II (a):$\mu \leq y$,
where $y=\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta+\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4(3\alpha+\beta-4\alpha\beta)\delta}$

It takes the form
$\left|a_3-\mu a_2^2\right| \leq \frac{\delta}{3\alpha+\beta-4\alpha\beta}$ (3.10)

Combining results (3.8) of subcase I (b) and (3.10) of subcase II (a), we obtain
$\left|a_3-\mu a_2^2\right| \leq \frac{\delta}{3\alpha+\beta-4\alpha\beta}$ $if w \leq \mu \leq y$ (3.11)

Where, $w=\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta-\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4(3\alpha+\beta-4\alpha\beta)\delta}$
and$y=\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta+\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4(3\alpha+\beta-4\alpha\beta)\delta}$

Subcase II (b):$\mu \geq y$,
where, $y=\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta+\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4(3\alpha+\beta-4\alpha\beta)\delta}$

Proceeding as in subcase I(a) , we get

$$\left|a_3-\mu a_2^2\right| \leq \frac{\delta}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[4\mu\delta-\frac{(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-3\alpha\beta+9\alpha^2\beta^2)\delta+\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}\right]\delta.$$

(3.12)

Hence the theorem is proved. The result is sharp in the sense that right hand side is the least upper bound of the result and extremal function exists.

Extremal function for first and third inequality is defined by $f_1(z) = z\{1 + \sqrt{f}(2\delta - g)z\}^{\frac{2\delta}{2\delta - g}}$
Extremal function for second inequality is defined by $f_2(z) = z(1 + \frac{1}{3\alpha+\beta-4\alpha\beta} z^2)^{\delta}$.
COROLLARY 3.2: Putting δ=1,$\alpha = 1, \beta = 0$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, if \mu \leq \frac{2}{3}; \\ \frac{1}{3}\ if \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ \mu - 1, if\ \mu \geq \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.
COROLLARY 3.3: Puttingδ=1,$\alpha = 0, \beta = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if \mu \leq \frac{1}{2}; \\ 1 if \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, if\ \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

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