LOCAL FRACTIONAL FOURIER TRANSFORM IN FRACTIONAL DIFFERENTIAL EQUATION

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Abstract: The purpose of this paper is to give applications of Local fractional Fourier transform with the help of which local fractional differential equations can be solved. Some properties of Local fractional Fourier transforms are obtained.

Keywords: Fractional Fourier Transform, Local fractional derivative, fractional differential equation.

Introduction: The applications of fractional transforms to generalized function have been done time to time and their properties have been studied by various mathematicians. Fourier transform Isa very powerful tool for problems in signal processing and other applications. The fractional Fourier transform was proposed by Namias and developed by McBrideit. Furthermore, it has been studied by many researchers and contributed. The fractional calculus has several applications in various fields of Mathematics as well as in real life situations, such as Abel's integral equation, visco-elasticity, capacitor theory, conductance of biological systems[5, 7].

The idea of fractional operators, fractional derivative, fractional geometry has long back history but fractional transform has been rediscovered in quantum mechanics, optics, signal processing as well as in pattern recognition. Now days many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science are effectively solved by fractional Fourier and fractional Hartley transforms.

The classical theory of local fractional calculus introduced by Kolwankar and Gangal [3]which becomes useful tool in the areas ranging from fundamental science to engineering.

The paper is organized as follows:

In section 2, We study some definitions which are useful for further developments. The section 3, is devoted for the definition of local fractional Fourier transform. In section 4, we prove some basic properties of local fractional Fourier transforms. In the last section, we obtain Local fractional Fourier Transform of some functions.

2Definitions of Fractional Derivatives and Fractional Integrals

Definition 2.1: (Grunwald-Letnikov): The Grunwald-Letnikov fractional derivative of order α of the function *f(x)* is defined as[7],

$$_{a}D_{x}^{\propto} f(x) = \lim_{n\to\infty}\{\frac{(\frac{x-a}{N})^{-\propto}}{\Gamma(-\propto)}\sum_{j=0}^{N-1} f(x-j[\frac{x-a}{N}])\}$$

Where, $\propto \in \mathbb{C}$.

Definition2.2:(Riemann-Lowville):

If f(x) ∈ C[a, b] and a < x < b then,

$$D_{a^+}^{\propto} f(x) = \frac{1}{\Gamma(1-\propto)}\frac{d}{dx}\int_a^x f(t)(x-t)^{-\propto}dt,$$

Where, $\propto \in (0,1)$

Definition 2.3:(M. Caputo (1967)):If f(x) ∈ C[a, b] and a < x < b then the Caputo Fractional derivative of order∝ isdefined as follows [7],

$$_{a}D_{x}^{\propto} f(x) = \frac{1}{\Gamma(n-\propto)}\int_a^x f^n(\tau)(x-\tau)^{n-\propto-1}d\tau ,$$

$$Where, \propto \in (n-1,n)$$

3Definitions of local fractional Fourier Transform

This section is devoted for the definition of Local Fractional Fourier Transform.

Definition 3.1: The complex number z = x + iy can also be written in the polar form asz = $re^{i\theta}$,so that fractional order of a complex number z can be defined as

$z^{\propto} = (re^{i\theta})^{\propto}, \propto \in (0,1]$.

The above definition can also have equivalent formula in the form of trigonometric function defined by the expression,

$$z^{\propto} = \left(\sqrt{x^2+y^2}\right)^{\propto}(\cos(\theta\alpha) + i\sin(\alpha\theta)),$$

$where, \propto \in (0,1]$.

Definition 3.2:

Local Fractional Fourier Transform:

The Local Fractional Fourier transform of function *f(t)* is defined as follows,

$$\widehat{f_\propto(\xi)} = \int_{-\infty}^{+\infty} f(t)e^{-2\pi it\xi^\propto}dt\,, \propto \in (0\,,1]$$

where as ∝→1, the Local fractional Fourier transform tends to be an ordinary Fourier Transform. To find the Local fractional Fourier transform of given function f(t) at a pointξ.Thesufficient condition for convergence is

$\int_{-\infty}^{+\infty}|f(t)|dt < \infty$.

Definition 3.3:

Inverse Local Fractional Fourier Transform:

The Local Fractional Fourier transform of function $\widehat{f(\xi)}$ is defined as follows,

$$f(t) = \int_{-\infty}^{+\infty} \widehat{f(\xi)}e^{2\pi it\xi^\propto}dt\,, \propto \in (0\,,1]$$

where as ∝→1, the Inverse Local fractional Fourier transform tends to be an ordinary Inverse Fourier Transform.

Definition 3.4:

Local Fractional Derivative:

Let, f : [a; b] →ℝbe any function, if the limit

$$D^\propto{}_\mp \lim_{x\to y\mp} \frac{d^\propto(f(x)-f(y))}{d\mp(x-y)^\propto}$$

Exist and is finite, then f is said have left(right)LFD[3]of order∝ at x = y.

4Properties of local Fractional Fourier Transform

In this section, we prove some basic properties of Local Fractional Fourier Transform

(i) If $g(t) = f(at)$ where $a > 0$ then

$$\widehat{g_\propto}(\xi) = \frac{1}{a}\widehat{f_\propto}\left(\frac{\xi}{a^{1/\propto}}\right)$$

Proof: - By definition (3.2) we get,

$$\widehat{g_\propto}(\xi) = \int_{-\infty}^{+\infty} f(at)e^{-2\pi it\xi^\propto}dt\,, a > 0$$

Put $at = x \to adt = dx$

$= \frac{1}{a}\int_{-\infty}^{+\infty} f(x)e^{-2\pi i\frac{x}{a}\xi^\propto}dx$

$= \frac{1}{a}\int_{-\infty}^{+\infty} f(x)e^{-2\pi ix(\frac{\xi}{a^{1/\alpha}})^\propto}dx \quad = \frac{1}{a}\widehat{f_\propto}\left(\frac{\xi}{a^{1/\propto}}\right)$

(ii) If $g(t) = f(t+\beta)$ where$\beta \in \mathbb{R}$ then,

$$\widehat{g_\propto}(\xi) = e^{2\pi i\xi\beta^\propto}\widehat{f_\propto}(\xi)$$

Proof: - By definition (3.2) we get,

$$\widehat{g_\propto}(\xi) = \int_{-\infty}^{+\infty} f(t+\beta)e^{-2\pi it\xi^\propto}dt$$

Put $t + \beta = x \to dt = dx$

$= \int_{-\infty}^{+\infty} f(x)e^{-2\pi i(x-\beta)\xi^\propto}dx$

$= e^{2\pi i\xi\beta^\propto}\widehat{f_\propto}(\xi)$

(iii) If $f(t)$ is a function which vanishes at ∓∞ then,$\widehat{f^{(n)}{}_\propto}(\xi) = (2\pi i\widehat{f(\xi)}^\alpha)^n\widehat{f_\propto}(\xi)$

Proof: - By definition (3.2) we get,

$$\widehat{f^{(n)}{}_\propto}(\xi) = \int_{-\infty}^{+\infty} f^{(n)}(t)e^{-2\pi it\xi^\propto}dt$$

Apply integration [8] to the above equation we get,

$$\widehat{f^{(n)}{}_\propto}(\xi) = e^{-2\pi it\xi^\propto}\int_{-\infty}^{+\infty} f^{(n)}(t)dt - \int \frac{d}{dt}(e^{-2\pi it\xi^\propto})f^{(n-1)}(t)dt$$

Continuing this process n times and using properties that $f(t)$ vanishes at ∓∞, we get

$$\widehat{f^{(n)}{}_\propto}(\xi) = (2\pi i\widehat{f(\xi)}^\alpha)^n\widehat{f_\propto}(\xi)$$

5Local Fractional Fourier Transform of some Functions

Consider the function which is define in the following way,

$$g(t) = \begin{cases} D^\propto(n+\sqrt{t}), & 0 \le t < \infty \\ 0 & , \quad otherwise \end{cases}$$

with initial condition $g(t) = n$ and$\propto = 1/2$

then by using the definition (2.2), (3.2) and (3.4)

$$\widehat{g_\propto}(\xi) = \int_{-\infty}^{+\infty} g(t)e^{-2\pi it\xi^\propto}dt$$

$= \int_0^{+\infty}[D^\propto(n+\sqrt{t})]e^{-2\pi it\xi^\propto}dt$

By using the given initial condition, we get

$\widehat{g_\propto}(\xi) = \int_0^{+\infty}[\lim_{t\to 0}\frac{d^{0.5}}{dt^{0.5}}(\sqrt{t})]e^{-2\pi it\xi^\propto}dt$

$= \int_0^{+\infty}[\lim_{t\to 0}(\frac{1}{\Gamma(0.5)}\frac{d^{0.5}}{dt^{0.5}}\int_a^t \tau^{0.5}(t-\tau)^{-0.5}dt)]e^{-2\pi it\xi^\propto}dt$

$= \int_0^{+\infty}[\lim_{t\to 0}(\frac{1}{\sqrt{\pi}}\frac{d^{0.5}}{dt^{0.5}}\int_a^t \tau^{0.5}(t-\tau)^{-0.5}dt)]e^{-2\pi it\xi^\propto}dt$

$= \int_0^{+\infty}[\lim_{t\to 0}\frac{1}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2}]e^{-2\pi it\xi^\propto}dt$

$= \frac{1}{4\pi i\xi^\propto}$

6Applications of Fractional Transforms

Consider the following fractional order differential equation,

$$0^{D^\propto}(y(t)) = f(t), 0 < \propto \le 1$$

With initial condition, $y(0) = 0$, where

$$f(t) = \begin{cases} e^{-t} & , \ 0 \le t < \infty \\ 0 & , otherwise \end{cases}$$

We solve above fractional order initial value problem by Riemann-Liouville definition offractional derivativewheree^{-t} is in Schwartz Space.

Apply Local FrFT on both sides of the above Fractional Differential Equation.

We get,

$$\frac{1}{\Gamma(1-\propto)}\int_{-\infty}^{\infty}[\frac{d}{dt}\int_{0}^{t}y(\tau)(t-\tau)^{-\propto}d\tau]\,e^{-2\pi it\xi^{\beta}}dt$$
$$=\int_{0}^{\infty}e^{-t-2\pi it\xi^{\beta}}dt$$

which gives us the following equation,

$$\frac{1}{\Gamma(1-\propto)}2\pi it\xi^{\beta}\int_{-\infty}^{\infty}[\int_{0}^{t}y(\tau)(t-\tau)^{-\propto}d\tau]\,e^{-2\pi it\xi^{\beta}}dt=\int_{0}^{\infty}e^{-t(1+2\pi i\xi^{\beta})}dt$$

We define the function as follows,

$$h(t-\tau)=\begin{cases}\frac{(t-\tau)^{-\propto}}{\Gamma(1-\propto)} & ,0<\tau<t\\ 0, & otherwise\end{cases}$$

and

$$\mathrm{H}(\tau)=\begin{cases}0, & \tau<0\\ 1, & \tau>0\end{cases}$$

Then by *Convolution* theorem

$$\int_{-\infty}^{\infty}[H(\tau)h(\tau)]*y(\tau)(t)e^{-2\pi it\xi^{\beta}}dt$$
$$=\frac{1}{1+2\pi it\xi^{\beta}}(2\pi it\xi^{\beta})^{-1}$$

$$\widehat{y_{\beta}}(\xi)\int_{0}^{t}t^{-\propto}\,e^{-t2\pi i\xi^{\beta}}dt=\frac{1}{1+2\pi it\xi^{\beta}}(2\pi it\xi^{\beta})^{-1}$$

$$\widehat{y_{\beta}}(\xi)\left(2\pi it\xi^{\beta}\right)^{\alpha-1}=\frac{1}{1+2\pi it\xi^{\beta}}(2\pi it\xi^{\beta})^{-1}$$

Then we get,

$$\widehat{y_{\beta}}(\xi)=\frac{1}{1+2\pi it\xi^{\beta}}(2\pi it\xi^{\beta})^{-\propto}$$

Now we define the functions as follows:

$$h(t-\tau)=\begin{cases}\frac{1}{\Gamma(\propto)}(t-\tau)^{-(1-\alpha)},0<\tau<t\\ 0, \quad otherwise\end{cases}$$

$$g(\tau)=\begin{cases}e^{-\tau}, & 0\leq\tau<\infty\\ 0, & otherwise\end{cases}$$

So that we get,

$$\widehat{h_{\beta}}(\xi)=(2\pi it\xi^{\beta})^{-\propto}$$

And

$$\widehat{g_{\beta}}(\xi)=\frac{1}{1+2\pi it\xi^{\beta}}$$

This gives us,

$$\widehat{y_{\beta}}(\xi)=\widehat{g_{\beta}}(\xi)\widehat{h_{\beta}}(\xi)$$

Applying inverse Local FrFT on both sides to the above equation and using the definition of convolution we get,

$$y(t)=\frac{1}{\Gamma(\propto)}\int_{0}^{t}e^{-\tau}(t-\tau)^{\propto-1}d\tau$$

This is the required solution of the Riemann-Liouville equation.

Conclusions

(i) We have solved fractional Differential Equation with help of Local Fractional Fourier Transform.

(ii) We prove some properties of Fractional Fourier Transform and calculate the Local FrFT of one function.

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