COMPOSITE MULTIPLICATION OPERATORS ON L^2-SPACES OF VECTOR VALUED FUNCTIONS

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Abstract: In this paper we characterize composite multiplication operators on L^2-Spaces of vector valued functions and also make an attempt to characterize boundedness, adjoint, isometry and partial isometry of these operators.

Keywords: composite multiplication operator, isometry, partial -isometry, adjoint of an operator.

Introduction: Let X be a non-empty set, C be the field of complex numbers and $V(X)$ be a vector space of complex valued functions on X under the pointwise operations of addition and scalar multiplication. Let T be a mapping of X into X such that $f \circ T$ is in $V(X)$ whenever f is in $V(X)$. Define the composition transformation C_T on $V(X)$ as $C_T f = f \circ T$ for every f in $V(X)$. If $V(X)$ has a Banach spacestructure and C_T is bounded, then C_T is called the composition operatoron $V(X)$ induced by T.Let $u : X \to C$ be a function such that M_u ,definedas $M_u f = u \cdot f$ for every f in $V(X)$ be a bounded linear operator on $V(X)$.Then the product $C_T M_u$ which becomes a bounded operator on $V(X)$ is called a composite multiplication operator.

Let $B(H)$ be the Banach algebra of allbounded operators on a Hilbert space H. If (X, Σ, μ) is a σ-finite measure space and $T: X \to X$ isa measurable transformation such that $C_T \in B(L^2(\mu))$, then in [1] R.K.Singh and D.C.Kumar has has the result proved, the measure μT^{-1}, defined as $\mu T^{-1}(E) = \mu(T^{-1}(E))$ for every E in Σ, is absolutely continuous with respect to the measure μ. Let f_0 denote the Radon-Nikodym derivative of μT^{-1} with respect to μ and if $C_T \in B(L^2(\mu))$, then in [2] R.K.Singh has proved that $C_T^* C_T = M_{f_0}$.

We consider the more general setting of weighted composition operators acting on $L^2(X, \Sigma, \mu, C^n)$, the Hilbert space of σ- measurable C^n-valued functions f for which $\|f(\cdot)\| \in L^2(X, \Sigma, \mu)$, where $\|\cdot\|$ represents the Euclidean norm. That is

$L^2(X, C^n) = \{ f / f : X \to C^n$ is a measurable and $\int_X \| f(\cdot) \|^2 d\mu < \infty \}$. Then is $L^2(X, C^n)$ is a Banach space under the norm,

$$\| f \| = \left(\int_X \| f(\cdot) \|^2 d\mu \right)^{\frac{1}{2}}$$

$L^2(X, C^n)$ is a Hilbert space under the inner product,

$$\langle f, g \rangle = \int_X \langle f(\cdot), g(\cdot) \rangle \, d\mu .$$

Let $u: X \to C^n$ be a vector- valued measurable function and let $T: X \to X$ be a non-singular measurable transformation. Then a bounded linear transformation

$M_{u,T} : L^2(X, C^n) \to L^2(X, C^n)$, defined by

$M_{u,T} = (u\, f) \circ T) = (u \circ T)\, (f \circ T)$

is called a composite multiplication operator induced by the pair (u, T). Where is a complex valued, Σ measurable function. In case $u = 1$ almost everywhere. $M_{u,T}$ becomes a composition operator, denoted by C_T.

A non-null set $A \in \Sigma$ is said to an atom if for every measurable set $F \subset A$ either $\mu(F) = 0$ or $\mu(F) = \mu(A)$.

A measure space (X, Σ, μ) is said to be non-atomic if for every non-null $A \in \Sigma$, there exists a non-null $F \in \Sigma$ with $F \subset A$ $\mu(F) < \mu(A)$. $A \in \Sigma$ is non-null $\mu(A) \neq 0$. When (X, Σ, μ) is a σ-finite non-atomic measure space.

We can easily to prove the following well known facts [3].

(a) Every σ-finite measure space (X, Σ, μ) can be decomposed into disjoint sets B and Z, such that μ is non-atomic over B and Z is at most countable union of atoms A_n of finite measure. So we can write X as follows:

$$X = B \cup (\cup_{n \in N} A_n)$$

(b) For each $f \in L^s(X, \Sigma, \mu)$, there exists two functions $f_1 \in L^p(X, \Sigma, \mu)$ and $f_2 \in L^q(X, \Sigma, \mu)$ such that $f = f_1 \cdot f_2$ and $\| f \|_s^s = \| f \|_p^p = \| f \|_q^q$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$

(c) Suppose $1 \le p < q < \infty$. If a Σ_0 - measurable set K, is non-atomic and such that $\mu(K) > 0$, there exists a function $g_0 \in L^p(X, \Sigma_0, \mu)$ with $\int_K |g_0|^p d\mu = \infty$.

Let X be a non-empty set and let Σ be a σ-algebra on X. Let μ and μT^{-1} be measures on Σ and $f_0 : X \to [0, \infty]$ be a measurable function.

Then the following are equivalent:

(i) μT^{-1} is absolutely continuous with respect to μ and f_0 is radon-Nikodym derivative of μT^{-1} with respect to μ.

(ii)For every measurable function $f : X \to [0, \infty]$, the equality

$$\int_X f \, d\mu T^{-1} = \int_X f_0 \, f \, d\mu$$

holds.

In the study of using conditional expectation of weighted composition operators on L^2. For each $f \in L^p(X, \Sigma, \mu)$ $1 \le p \le \infty$, there exists a unique $T^{-1}(\Sigma)$-measurable function $E(f)$ such that $\int_A g f \, d\mu = \int_A g E(f) \, d\mu$ for every $T^{-1}(\Sigma)$-measurable function g for which the left integral exists. The function $E(f)$ is called the conditional expectation of f with respect to the subalgebra $T^{-1}(\Sigma)$. As an operator of L^p, E is the projection onto the closure of range T and E is the identity on $L^p(\mu)$, $p \ge 1$ if and only if $T^{-1}(\Sigma) = \Sigma$. Detailed discussion of E is found in [4 , 5].

During the last thirty years several authors, define $W_{u,T} = M_u C_T = u \cdot (f \circ T)$, have studied the properties of various classes of weighted composition operators on L^2 Spaces. The several results on weighted composition operators have been studied by R.K.Singh and D.C.Kumar [1]. Boundedness of the composition operators in $L^p(X, \Sigma, \mu)$, ($1 \le p < \infty$) spaces, where the measure spaces are σ-finite, appeared already in [6]. Also boundedness of weighted operators on $C(X, E)$ has been studied in [7]. Many classes of weighted composition operators on some function spaces are considered in [8,9, 10 ,11]. In this paper we to study composite multiplication operators on vector valued L^p-spaces.

Composite multiplication operators

Theorem 2.1

Let $M_{u,T} : L^2(X, C^n) \to L^2(X, C^n)$ be a linear transformation. Then $M_{u,T}$ is bounded if and only if $J \in L^\infty(X, C^n)$, where $J = f_0 u^* u$

Proof:

For any $f \in L^2$, we have

$$\| M_{u,T} \|_2^2 = \int \langle u f \circ T, u f \circ T \rangle d\mu$$
$$= \int \langle E(u f \circ T), E(u f \circ T) \rangle d\mu$$
$$= \int \left\langle f_0 u f, u f \right\rangle d\mu$$
$$= \int \left\langle f_0 u^* u f, f \right\rangle d\mu$$

Setting $J = f_0 u^* u$, it follows that $| M_{u,T} | = M_{\sqrt{J}}$, where $M_{\sqrt{J}}$ is the operator on L^2 defined by

$$M_{\sqrt{J}} f = \sqrt{J} f$$

Also, for any $f, g \in L^2$, we have

$$\left\langle M_{u,T} f, g \right\rangle = \int \left\langle M_{u,T} f, g \right\rangle d\mu$$
$$= \int \langle u f \circ T, g \rangle d\mu$$
$$= \int \langle E(u f \circ T), E(g) \rangle d\mu$$
$$= \int \left\langle f, u^* f_0 E(g) \circ T^{-1} \right\rangle d\mu$$
$$= \int \left\langle f, M^*_{u,T} g \right\rangle d\mu$$

and it follows that

$$M^*_{u,T} g = u^* f_0 E(g) \circ T^{-1} \text{ for every } g \in L^2$$

In the next theorem, we characterize the boundedness of composite multiplication operator for atomic measure spaces.

Theorem 2.2

$M_{u,T} \in C(L^2(X, C^n))$ if and only if $J : N \to C^{n \times n}$ is a bounded function, where

$$J(n) = \sum_{m \in T^{-1}(\{n\})} \frac{\mu(m) u^*(n) u(n)}{\mu(n)}.$$

Proof:

For any $f \in L^2(N, C^n)$, consider

$$\left\| M_{u,T} \right\|^2 = \sum_{n=1}^{\infty} \langle (u f \circ T)(n), (u f \circ T)(n) \rangle \mu(n)$$

$$= \sum_{n=1}^{\infty} \langle u(T(n))\, f(T(n)), u(T(n)) f(T(n)) \rangle \mu(n)$$

$$= \sum_{n=1}^{\infty} \sum_{m \in T^{-1}(\{n\})} \langle u(n)\, f(n), u(n)\, f(n) \rangle \mu(m)$$

$$= \sum_{n=1}^{\infty} \sum_{m \in T^{-1}(\{n\})} \left\langle u^*(n) u(n)\, f(n), f(n) \right\rangle \mu(m)$$

$$= \sum_{n=1}^{\infty} \left\langle \sum_{m \in T^{-1}(\{n\})} \frac{\mu(m) u^*(n) u(n)}{\mu(n)} f(n), f(n) \right\rangle \mu(n)$$

$$= \sum_{n=1}^{\infty} \langle J(n)\, f(n), f(n) \rangle \mu(n)$$

$$= \left\| M_{\sqrt{J}} \right\|^2$$

Hence $M_{u,T}$ is a bounded operator if and only if $J : N \to C^{n \times n}$ is a bounded function.

Theorem 2.3

Let $M_{u,T} \in C(L^2(X, C^n))$.

Define $(A g)(n) = \frac{1}{\mu(n)} \sum_{m \in T^{-1}(\{n\})} \mu(m) u^*(n)\, g(m)$ for every $g \in L^2(N, C^n)$.

Proof:

For any $f, g \in L^2(N, C^n)$, consider

$$\langle M_{u,T}\, f, g \rangle = \sum_{n=1}^{\infty} \langle (u f \circ T)(n), g(n) \rangle \mu(n)$$

$$= \sum_{n=1}^{\infty} \langle u(T(n))\, f(T(n)), g(n) \rangle \mu(n)$$

$$= \sum_{n=1}^{\infty} \sum_{m \in T^{-1}(\{n\})} \langle u(n)\, f(n), g(m) \rangle \mu(m)$$

$$= \sum_{n=1}^{\infty} \sum_{m \in T^{-1}(\{n\})} \left\langle f(n), u^*(n)\, g(m) \right\rangle \mu(m)$$

$$= \sum_{n=1}^{\infty} \left\langle f(n), \sum_{m \in T^{-1}(\{n\})} \frac{\mu(m) u^*(n) g(m)}{\mu(n)} \right\rangle \mu(n)$$

$$= \sum_{n=1}^{\infty} \left\langle f(n), (M^*_{u,T}\, g)(n) \right\rangle \mu(n)$$

$$= \langle f, A g \rangle$$

Hence $M^*_{u,T} = A$.

Theorem 2.4

Let $M_{u,T} \in C(L^2(X, C^n))$. Then $M_{u,T}$ is a partial isometry if and only if J is an idempotent.

Proof:

Suppose $M_{u,T}$ is a partial isometry. Then

$$M_{u,T} = M_{u,T}\, M^*_{u,T}\, M_{u,T}$$

and therefore

$$M^*_{u,T}\, M_{u,T} = M^*_{u,T}\, M_{u,T} M^*_{u,T}\, M_{u,T}$$

$$M_J = M_{J^2}$$

Hence we conclude that J is an idempotent.

Conversely, if J is an idempotent mapping then, since $\operatorname{Ker} M_{u,T} = \operatorname{Ker} M_J$,

So for any $f \in (\operatorname{Ker} M_{u,T})^{\perp} = \overline{\operatorname{ran} M_J}$, we have,

$$\left\langle M^*_{u,T}\, M_{u,T}\, f, g \right\rangle = \int_X \langle u f \circ T, u g \circ T \rangle \, d\mu$$

$$= \int_X \langle E(u f \circ T), E(u g \circ T) \rangle \, d\mu$$

$$= \int_X \left\langle u^* u\, f_0 f, g \right\rangle d\mu$$

$$= \int_X \langle J f, g \rangle \, d\mu$$

$$= \langle J f, g \rangle$$

Hence $M_{u,T}$ is a partial isometry.

Theorem 2.5

Let $M_{u,T} : L^2(X, C^n) \to L^2(X, C^n)$ be a bounded operator. Then $M_{u,T}$ is an isometry if and only if $J^{\frac{1}{2}}(x)$ is an isometry for μ - almost all $x \in X$.

Proof:

The proof follows from the equality,

$$\left\| M_{u,T}\, f \right\| = \left\| M_{\sqrt{J}}\, f \right\|$$

for every $f \in L^2(X, C^n)$.

Theorem 2.6

Let $M_{u,T} \in C(L^2(X, C^n))$. Then $M_{u,T}$ is an idempotent operator if and only if $u \circ T \cdot u \circ T^2 = u \circ T$ and $T^2 = T$ on $\operatorname{supp}\, u \circ T \cap \operatorname{supp}\, u \circ T^2$.

Proof:

Suppose $M_{u,T}$ is an idempotent operator. Then for $e_k \in C^n$, we have for any $E \in \Sigma$ with $\mu(E) < \infty$,

$$M_{u,T}\, M_{u,T}\, (\chi_E\, e_k) = M_{u,T}(\chi_E\, e_k)$$

which implies that

$u \circ T \cdot u \circ T^2(\chi_{(T^2)^{-1}(E)} e_k) = u \circ T(\chi_{T^{-1}(E)} e_k)$ Hence $u \circ T \cdot u \circ T^2 = u \circ T$ and $T^2 = T$ on $\operatorname{supp} u \circ T \cap \operatorname{supp} u \circ T^2$.

The converse is easy to prove.

Example: 2.1

Let $X = [0,1]$ and

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ -2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

and $u(x) = 2x$ for every $x \in X$. Then $f_0 = 1$ (a.e), so

$$\| f_0 u^* u \|_2^2 = \int_0^1 | f_0 u^* u |^2 \, d\mu = \frac{4}{3}$$

Hence $M_{u,T}$ is a bounded operator.

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