DYNAMICAL BEHAVIOR IN DISCRETE TWO SPECIES PREDATOR-PREY SYSTEM

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Abstract: In this paper, a discrete two species predator-prey system with Holling type II is obtained. The possible equilibrium points are observed and their local stability analyzed. The bifurcation diagrams and phase portraits have been obtained for selected rang of different parameters. As some parameters varied, the model exhibit chaos as long as time behavior. Also, the fractal dimensions are presented.

Keywords: discrete system, stability analysis, bifurcation analysis.

Introduction: The Lotka-Volterra equations describe an ecological predator- prey (or parasite- host) model and it is the simplest model of population dynamics reveal the delicate balance that exists in almost all ecological systems.

Actually, Vito Volterra proposed a prey population fluctuate [1]. After that Holling suggested three types of functional response to describe the predation in different species [2].Hus and Hwang [3] impose Hollinge type II in the continuous predator-prey system and study their global stability. Then, Agiza et al [4] investigated a discrete two species predator-prey system with Holling type II as another way to understand the complex problem of computation. This discrete system exhibit rich dynamics compared with the continuous one. In the present work, we modified the system in [4] to become more realistic by adding the death rate in predator population.

One of the real life applications of the two species predator- prey system is the Lynx and its prey the snowshoe Hare study documented by the Hudson Bay Company for the time interval 1845–1935 [5]. The extension of discrete two species predator-prey system to cover the Holling type II had a little attention in the discrete case till now, due to its complexities. Therefore, the present work aims to shed more light on this subject through analyzing the dynamic complexities in a discrete two species predator-prey system with the Hollings type II functional response. That is, we shall focus our attention on analyzing how the Holling type II response [2] affects the dynamic complexities of prey–predator interactions.

This paper is organized as follows: in Section 2, the discrete two species predator-prey system with Holling type II is formulated and the possible equilibrium points are found. In Section 3, the stability analyses of the existence three equilibrium points are observed. In Section 4, some examples of different values of parameters derived and discussed as numerical simulation of the analytic results. Fractal dimensions were presented in Section 5. Finally, Section 6 draws the conclusion.

Mathematical model:

From [4], we see that the discrete two species predator-prey map with Holling type II has the following difference equations system:

$$\begin{cases} x_{n+1} = ax_n(1 - x_n) - \dfrac{rx_n y_n}{1 + \epsilon x_n} \\ y_{n+1} = \dfrac{mx_n y_n}{1 + \epsilon x_n} \end{cases} \tag{1}$$

Where a, r, ϵ and m greater than zero.

The map given by equation (1) is noninvertible in the plane and for more realistic and interesting system suitable to the predator-prey interaction we imposed the nature death rate in the predator population to become as following system:

$$\begin{cases} x_{n+1} = ax_n(1 - x_n) - \dfrac{rx_n y_n}{1 + \epsilon x_n} \\ y_{n+1} = -dy_n + \dfrac{mx_n y_n}{1 + \epsilon x_n} \end{cases} \tag{2}$$

Where r and m are the half saturation parameters. While the positive parameters a, ϵ and d represent the prey intrinsic rate, the limitation of the growth velocity of the predator with increase in the number of prey and predator death rate, respectively.

If $a > 1$ and $m > \max\{\epsilon(1 + d), (1 + d)[a(1 + \epsilon) - \epsilon]/a - 1\}$ then system (2) has the following equilibrium points:

1. $(x, y) = (0,0)$ is the trivial equilibrium point according to the extinction of prey and predator species.
2. $(x, y) = (\frac{a-1}{a}, 0)$ is the axial equilibrium point according to the present of prey absence of predator population.
3. $(x, y) = (x^*, y^*)$ is the unique positive equilibrium point according to the interaction between the prey and predator species, where $x^* = 1 + d/m - \epsilon(1 + d)$ and $y^* = [a(1 - x^*) - 1](1 + \varepsilon x^*)/r$.

Now, we investigate the local behavior of the model (2) around each of the above equilibrium points. The local stability analysis of the model (2) can be studied by computing the variation matrix corresponding to each equilibrium point. The Jacobian matrix of Eq. (2) at the state variable is given by:

$$J(x,y)=\begin{bmatrix} a(1-2x)-\frac{ry}{(1+\epsilon x)^2} & -\frac{rx}{1+\epsilon x} \\ \frac{my}{(1+\epsilon x)^2} & \frac{mx}{1+\epsilon x}-d \end{bmatrix} \quad (3)$$

and its characteristic equation is:

$F(\lambda)=\lambda^2+Tr\big(J(x,y)\big)\lambda+Det(J(x,y))$ (4)

Where Tr is the trace and Det is the determinant of the Jacobian matrix $J\,(x,y)$ which are defined as:

$$Tr\big(J(x,y)\big)=a(1-2x)-d+\frac{mx}{1+\epsilon x}-\frac{ry}{(1+\epsilon x)^2}$$

and

$$Det\big(J(x,y)\big)=ad(2x-1)+\frac{amx}{1+\epsilon x}-\frac{dry}{(1+\epsilon x)^2}$$

Stability analysis: In this section, we will observe the stability conditions of the existence three equilibrium points for the system(2). The present section aims to shed more light on the positive equilibrium points by analyzing the dynamic complexities. Due to that, this section has the following two sub sections.

Stability (0,0) and$(\frac{a-1}{a},0)$**:**

This sub section contains the local stability conditions of the origin equilibrium point (0,0) and the exits axial point$(\frac{a-1}{a},0)$ which observed in the following theorems:

Theorem 1: The origin equilibrium point (0,0) is locally asymptotic stable if $a,b<1$ and unstable otherwise.

Theorem 2: The axial equilibrium point $(\frac{a-1}{a},0)$ is:

1. Stable if $1<a<3$ and $m<\frac{(1+d)[a+\epsilon(a-1)]}{a-1}$;
2. Unstable otherwise.

Proof: According to equation(4)it is easy to verify that the eigenvalues of $J(\frac{a-1}{a},0)$ are $\lambda_1=2-a$ and $\lambda_2=m(a-1)-d[a+\epsilon(a-1)]/a+\epsilon(a-1)$.

Therefore, for $1<a<3$ and $m<\frac{(1+d)[a+\epsilon(a-1)]}{a-1}$ then $|\lambda_i|<1$ for all $i=1,2$ and hence p_1 is sink. While, for all other sets of parameters $(\frac{a-1}{a},0)$ is an unstable point.

The axial equilibrium point $(\frac{a-1}{a},0)$ can undergo flip bifurcation when the parameters vary and the center manifold of the system(2) at $(\frac{a-1}{a},0)$ is $y=0$ which is restricted to the center manifold of the logistic equation [6]. In this case the predator population becomes extinction and the prey population passes through periodic doubling bifurcation to chaos in sense of Li-York by varying the bifurcation parameter a[7].

Stability(x^*,y^*)**:**

Before we set the local stability conditions near the interior point(x^*,y^*), we recall the Jury's conditions as follow:

The positive equilibrium point(x^*,y^*) is locally asymptotic stable if and only if the following conditions hold:

1. $f(1)=1-tr(x^*,y^*)+\det(x^*,y^*)>0$;
2. $f(-1)=1+tr(x^*,y^*)+\det(x^*,y^*)>0$;
3. $|\det(x^*,y^*)|<1$.

Theorem 3: The positive equilibrium point(x^*,y^*) is locally asymptotic stable if and only if the following inequalities hold:

$m>\max\{\epsilon(1+d),\frac{(1+d)[a(\epsilon+1)-\epsilon]}{a-1}\}$ (5)

$a>\frac{2ry^*(1+\epsilon x^*)-rmx^*y^*-2(1+\epsilon x^*)^3}{2(1-2x^*)(1+\epsilon x^*)^3}$ (6)

$a<\frac{ry^*(1+\epsilon x^*)-rmx^*y^*}{(1-2x^*)(1+\epsilon x^*)^3}$ (7)

Obviously, the inequalities $(5)-(7)$ are equivalent to the above Jury's conditions so that if they are satisfying then guarantee our positive equilibrium point is stable.

For the positive equilibrium point(x^*,y^*), the roots of its Jacobian matrix are:

$\lambda_{1,2}=\frac{-Tr(J(x^*,y^*))\pm\sqrt{\Delta}}{2}$ (8)

Where

$\Delta=(Tr(J(x^*,y^*)))^2-4Det(J(x^*,y^*))$,

$Tr\big(J(x^*,y^*)\big)=1+a(1-2x^*)-\frac{ry^*}{(1+\epsilon x^*)^2}$,

and

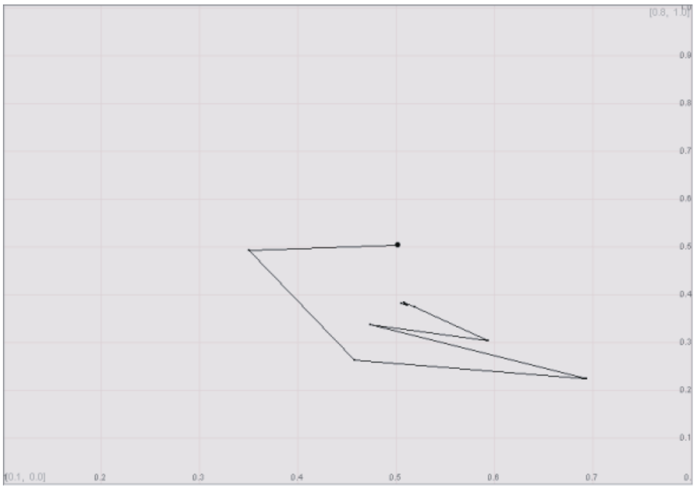
$Det\big(J(x^*,y^*)\big)=a(1-2x^*)-\frac{ry^*}{(1+\epsilon x^*)^2}+\frac{rmx^*y^*}{(1+\epsilon x^*)^3}$.

The eigenvalues $\lambda_{1,2}$ are real for λ_R and $|\lambda_{1,2}|<1$ if $\Delta>0$ and$F(1)>0$ which implies the conditions that determine the domains of the values of a,ϵ and d such that (x^*,y^*) is stable equilibrium point. So, without loss the generality we have the following theorem:

Theorem 4: if $\Delta<0$ and $Det\big(J(x^*,y^*)\big)=1$ then the system(2) may undergo Hopf bifurcation at(x^*,y^*). Moreover, an attractive invariant closed curve bifurcates from (x^*,y^*) for$Det\big(J(x^*,y^*)\big)>1$.

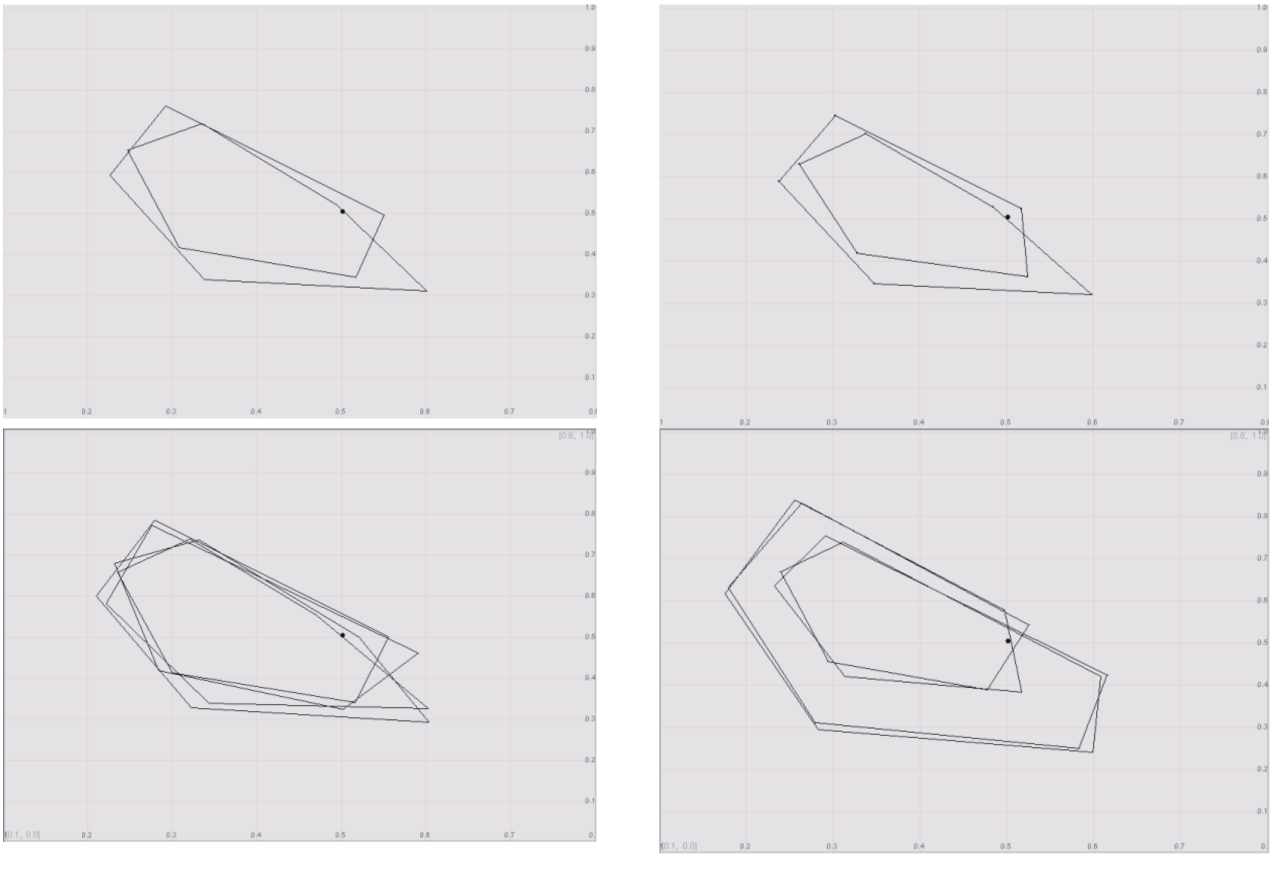
Numerical simulations: In this section, our aim is to present the numerical simulation to explain the above theoretical results by the bifurcation diagrams and phase portraits for system (2) around the positive equilibrium (x^*,y^*) and show the new interesting complex dynamical behavior. As the predator death rate d varying in the interval$[0,0.823]$, the system (2) exhibit different phenomena which presented in the following examples:

Example 1: For the values $a=4.1,\epsilon=0.22,r=3.0,m=3.5$ and$d=0.6$ the computation yield$(x^*,y^*)=(0.5082,0.3765)$. The phase portrait in figure (1) shows an attractive trajectory toward the point(x^*,y^*) .



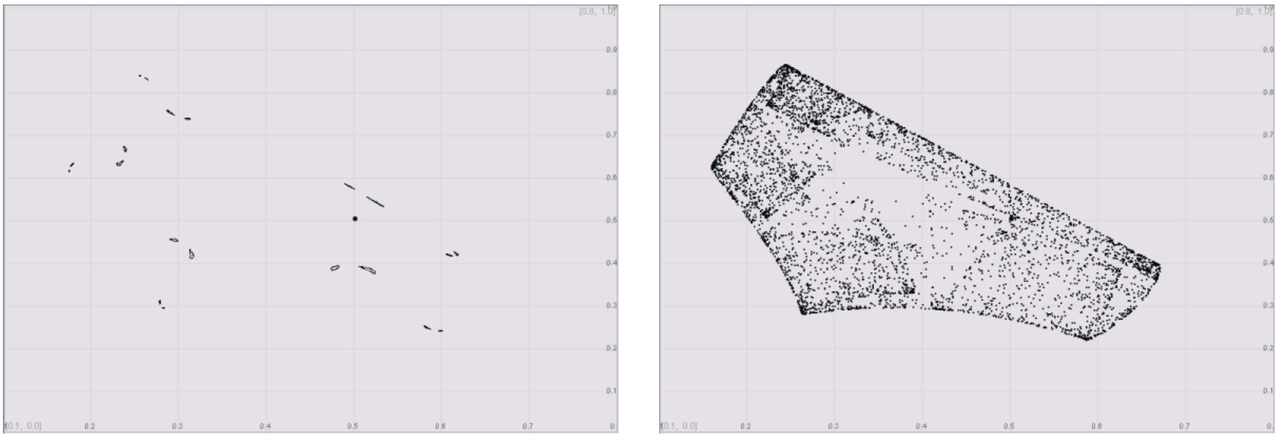
Figure(1): The phase portrait of system (2) when $d = 0.6$.

Example 2: If we fixed the values of the parameters a, r, ϵ and m as in the example (1) and crossing the parameter d from the values 0.2,0.18,0.155 and 0.112 that imply the variant closed curves which created around the equilibrium point (x^{*}, y^{*}) with period $-9, -10, -21$ and period -20, respectivally. See the figure(2)



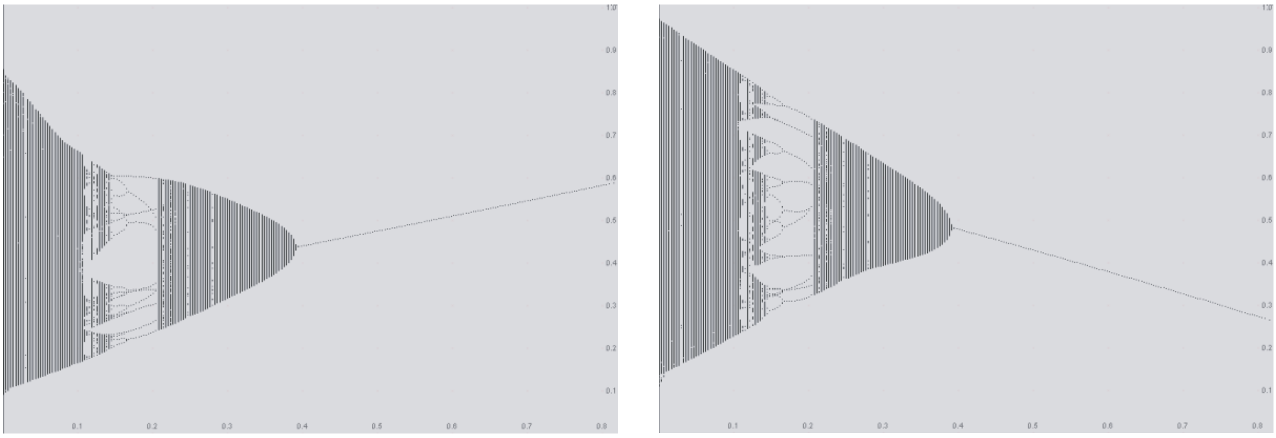
Figure(2): The phase portrait of system (2) when $d = 0.2, 0.18, 0.155$ and0.112 , respectively.

Example 3: The system (2) with the same values as in example (1) and the initial value $(0.5, 0.5)$ produce the strange attractor when $d = 0.1$and0.09 , respectively. (see figure(3))



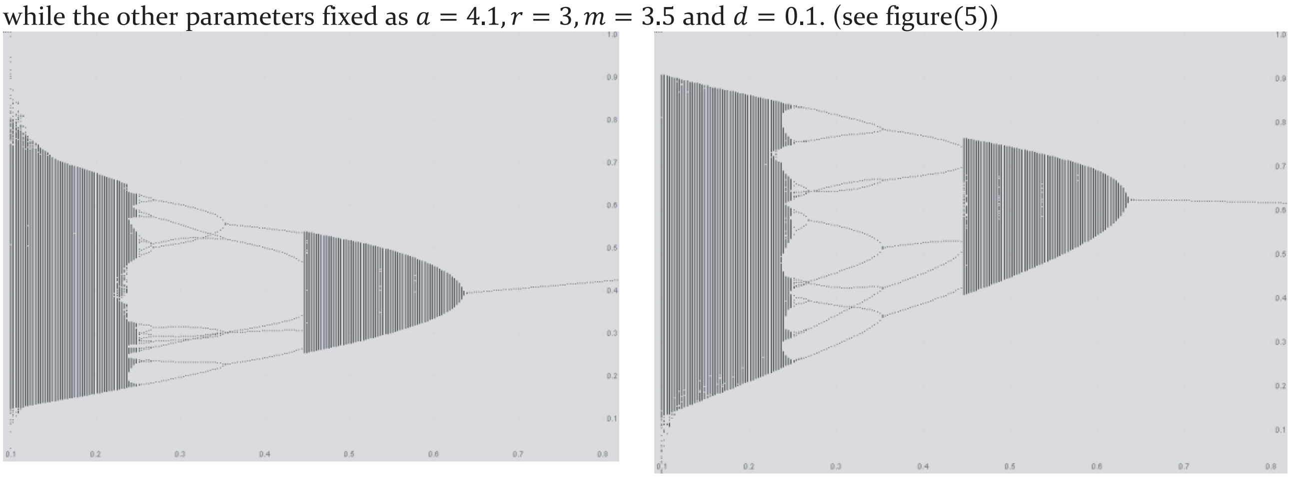
Figure(3): The phase portrait of system (2) when $d = 0.1$ and0.09 , respectively.

Now, the bifurcation diagrams will provide the information about abrupt change in the dynamics behavior and the values of parameters at which changes occur (see the above examples).Also, they will give the information about the dependence of the dynamics on certain parameters. In figure (4) we plotted the bifurcation diagrams as d changed inside the interval $[0, 0.82]$ in which the rest parameters are $a = 4.1, \varepsilon = 0.22, r = 3.0$ and $m = 3.5$ with the initial value$(0.5, 0.5)$.



Figure(4): The bifurcation diagrams for $d - prey$ and$d - predator$, respectively.

Furthermore, if the parameterε vary in the closed interval $[0.09, 0.82]$ then the system (2) tend to the chaotic while the other parameters fixed as $a = 4.1, r = 3, m = 3.5$ and $d = 0.1$. (see figure(5))



Figure(5): The bifurcation diagrams of $\epsilon - prey$ and$\epsilon - predator$, respectively.

Fractal dimensions: The other way to characterize the strange attractors is by using fractal dimensions. To do this, we will examine the important characteristics of neighboring chaotic orbits to see how rapidly they separate each other. This separation is quantified by using the concept of Lyapunov exponents[8]. These exponents represent a dynamic measure of chaos that average the separation of the orbits of nearby initial conditions as system moves forward in time. The Lyapunov dimension $[9-11]$ is defined by using Lyapunov exponents as follows:

$$L_d = d + \frac{\sum_{i=1}^{i=j} \Lambda_i}{|\Lambda_i|} \qquad (9)$$

With $\Lambda_1, \Lambda_2, \dots \Lambda_n$, where d is the largest integer such that $\sum_{i=1}^{i=j} \Lambda_i \geq 0$ and $\sum_{i=1}^{i=j+1} \Lambda_i < 0$.

The system (2) is a two-dimension map which has the Lypunov dimension in the form:

$$L_d = 1 + \frac{\Lambda_1}{|\Lambda_2|} \qquad (10)$$

With $\Lambda_1 > 0 > \Lambda_2$.

The two Lypunov exponents are estimated and are found to be $\Lambda_1 \approx 0.0941$ and $\Lambda_2 \approx -0.4651$ for$\epsilon = 0.7$. The above results show that the present system has fractal dimension in the following form:

$$L_d = 1 + \frac{0.0941}{0.4651} \approx 1.2023$$

So, one can deduce that our system is very complicated and rich system.

Conclusion: This paper dealt with a discrete two species predator-prey system with Holling type II. All equilibrium points are found and their stability is investigated. The numerical simulation analyzes our system by means of phase portrait, bifurcation diagrams. The period$-9, -10, -21$ and-20appeared with some values of the predator death rate$d$. The present system is exhibit very interesting behavior which is useful for the biologists as well as mathematicians.

References:

1. V. Voltera, Opere matematiche: mmemorie e note, vol. V, Acc. Naz. dei Lincei, Roma, Cremon, 1962.
2. C. Jaya Subba Reddy, S.Mallikarjunarao, T. Mahesh Kumar, Left Multiplicative Generalized Derivations on Semi Prime; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 859-861
3. C.S. Holling, The functional response of predator to prey density and its role in mimicry and population regulation, Mem. Ent. Soc. Canada, 45(1965) 1-60.
4. S.B. Hsu, T.W. Hwang, Global stability for a class predator–prey system, SIAM J. Appl. Math. 55 (1995) 763–783.
5. H.N. Agiza, E.M. ELabbasy, H. EL-Metwally, A.A. Elsadany, Chaotic dynamics of a discrete prey–predator model with Holling type II, Nonlinear Analysis: Real World Applications 10 (2009) 116–129.
6. *Krishna Kumar Singh, Lakkarajugovinda,* A Faster Approximation Algorithm for independent Set ; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 439-441
7. L. Edelstein-Keshet, Mathematical Models in Biology, Random House, New York, 1988.
8. J.D. Murray, Mathematical Biology, Springer-Verlag, Berlin, 1998.
9. S.N. Elaydi, An Introduction to Difference Equations, Springer-Verlag Publishers, 1996.
10. X. Liu, D. Xiao, Complex dynamic behaviors of a discrete-time predator–prey system, Chaos Solitons Fractals 32 (2007) 80–94.
11. V.I. Oseledec, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, Trans. Moscow Math. Soc. 19(1968) 197–231.
12. Parvinder Singh, Homotopy theory and Contractible Spaces; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 412-414
13. J.L. Kaplan, Y.A. Yorke, A regime observed in a fluid flow model of Lorenz, Comm. Math. Phys. 67 (1979) 93–108.
14. J.H.E. Cartwright, Nonlinear stiffness, Lyapunov exponents, and attractor dimension, Phys. Lett. A 264 (1999) 298–304.

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