COMMON FIXED POINT THEOREM USING COMPARISION FUNCTION IN CONE b-METRIC SPACES

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Abstract: In this paper, a common fixed point theorem using comparison function is proved in cone b-metric space which is the generalisation of the result of W. Shatanawi *et.al* in [8] proved in b-metric space.

Keywords: Cone b-metric space, common fixed point theorem, comparison function.

Introduction: The concept of b-metric spaces was introduced by Bakhtin in 1989 and the cone metric spaces was introduced by Huang and Zhang[1] in 2007. Abbas and Jungck[2] proved common fixed point theorems in cone metric spaces. For the properties and examples of cone metric spaces, see[3-5]. Hussain and shah[6] introduced the notion of a cone b-metric spaces generalising both b-metric spaces and cone metric spaces for normal cones. Huang and Xu[7] proved the same results without the assumption of normality. W.Shatanawi[8] proved the common fixed point theorem in b-metric spaces using comparision function. For the examples of comparision function, see[9]. In this paper, we extend the result of common fixed point theorem in b-metric spaces to cone b-metric spaces by means of a comparision function.

Preliminaries:

Definition 1.1: Let E be a Banach space and $P \subseteq E$. P is called a cone if it satisfies the following properties:

(i) P is closed, non-empty and $P \neq \{\theta\}$, where θ is the zero element of E.

(ii) $a, b \in R; a, b \geq 0; x, y \in P implies ax + by \in P$.

(iii) $x \in P and -x \in P implies x = \theta$.

Definition 1.2 For a given cone P, define a partial ordering $\leq$ w.r.t. P by

(i) $x \leq y iff y - x \in P$

(ii) $x < y implies x \leq y and x \neq y$

(iii) $x \ll y implies y - x \in \mathrm{Int}P, where IntP$ denotes interior of P.

Throughout this paper, E stands for Banach space and P for solid cone. For the properties and examples of cone metric spaces, see[3-5].

Definition 1.3 Let X be a non-empty set and E be a real Banach space with cone P. A vector valued function $d: XxX \to E$ is said to be coneb-metric space on X with a real constant s ≥ 1 if the following conditions are satisfied for all $x, y, z \in X$:

(i) $\theta \leq d(x, y) i.e., d(x, y) \in P$

(ii) $d(x, y) = \theta iff x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq s[d(x, z) + d(z, y)]$

The pair (X, d) is said to be cone b-metric space.

Clearly, the class of cone b-metric space is larger than the class of cone metric space.

Example 1.4 Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\} \subset E$, $X = R$ and $d: XxX \to E$ such that $d(x, y) = (|x - y|^p, \alpha|x - y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space, but not a cone metric space.

Definition 1.5: Let(X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X. Then

(i) $x_n \to x$ whenever for every $c \in E$ with $\theta \ll c$, there is a natural number N such that

$$d(x_n, x) \ll c \forall n \geq N.$$

(ii) $\{x_n\}$ is a Cauchy Sequence if for every $c \in E$with $\theta \ll c$ there is a natural number N such that

$$d(x_n, x_m) \ll c \forall n, m \geq N.$$

(iii) (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent in X.

The following lemmas are often used in proving our main result.

Lemma 1.6: Let(X, d) be a cone b-metric space over a real Banach space E with a cone P. The following properties hold for all $x, y, z \in E$.

(i) $If x \leq y and y \ll z then x \ll z$.

(ii) $If \theta \leq x \ll c for all c \in IntP, then x = \theta$.

(iii) $If x \leq \lambda x where x \in P and 0 \leq \lambda < 1 then x = \theta$.

(iv) $Let x_n \to \theta in E and let \theta \ll c. Then \exists n_0 \in N such that x_n \ll c for each n \geq n_0$.

Definition 1.7 : Let (X, d) and (X', d') be two cone b-metric spaces with real constants s and s' respectively. A mapping $f: X \to X'$ is said to be continuous if for each sequence $\{x_n\}$in X, which converges to $x \epsilon X$ w.r.t d, then $\{fx_n\}$ converges to fx w.r.t d'.

Definition 1.8: Let $s \geq 1$ be a constant. A mapping $\varphi: P \to P$ is called a comparision function over P with base $s \geq 1$, if it satisfies the following:

(i) φ is non-decreasing

(ii) $\lim_{n \to \infty} \varphi^n(t) = \theta$ for all $\theta < t$.

Remark 1.9 If φ is a comparision function, then$\varphi(t) < t$ for all $\theta < t$.

Main Result:

Theorem 2.1: Let (X, d) be a complete cone b-metric space and with a real constant $s \geq 1. f, g$ be two self-maps on X. Suppose there is a constant $L < \frac{1}{1+s}$ and a comparision function φ over the cone $P \subseteq E$ satisfying the inequality

$sd(fx, gy) \leq \varphi(\max\{sd(x, fx), sd(y, gy), L[d(x, gy) + d(fx, y)]\})$for all $x, y \in X$ (1)

If any of the mappings $forg$ is continuous, the $fandg$ have unique common fixed point.

Proof Let $x_0 \in X$. Define the sequence $\{x_n\}$ as $x_{2n+1} = fx_{2n}$and $x_{2n+2} = gx_{2n+1} foralln \in N$.

First we shall prove the case if $x_n = x_{n+1}$for some $n \in N$.

If $n = 2k$, then $x_{2k} = x_{2k+1}$. Using (1),

$sd(x_{2k+1}, x_{2k+2}) = sd(fx_{2k}, gx_{2k+1})$

$\leq \varphi(\max\{sd(x_{2k}, x_{2k+1}), sd(x_{2k+1}, x_{2k+2}), L[d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})]\})$

$\leq \varphi(\max\{sd(x_{2k+1}, x_{2k+2}), L[sd(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})]\})$

$\leq \varphi(\max\{sd(x_{2k+1}, x_{2k+2}), L[sd(x_{2k+1}, x_{2k+2})]\})$

$= \varphi(sd(x_{2k+1}, x_{2k+2}))$

Using Remark 1.9, if $\theta < d(x_{2k+1}, x_{2k+2})$ then $sd(x_{2k+1}, x_{2k+2}) \leq \varphi(sd(x_{2k+1}, x_{2k+2})) < sd(x_{2k+1}, x_{2k+2})$, a contradiction.

Hence $d(x_{2k+1}, x_{2k+2}) = \theta$. $i.e.,\ \ x_{2k+1} = x_{2k+2}$.

Therefore,

$x_{2k} = x_{2k+1} = x_{2k+2} i.e., x_{2k} = fx_{2k} = gx_{2k}$ implies x_{2k} is the common fixed point of $fandg$.

Similarly if $n = 2k + 1$, it can be seen that x_{2k+1} is the common fixed point of $fandg$.

Now we prove the case when $x_n \neq x_{n+1}$ for all $n \in N$ in several steps.

Step I: To prove $sd(x_n, x_{n+1}) \leq \varphi(sd(x_{n-1}, x_n))$ for all $n \in N$. (2)

We prove this for $n = 2k$ and then for $n = 2k + 1$ where $k \in N$.

If $n = 2k$, then

$sd(x_{2k+1}, x_{2k}) = sd(fx_{2k}, gx_{2k-1})$

$\leq \varphi(\max\{sd(x_{2k}, x_{2k+1}), sd(x_{2k-1}, x_{2k}), L[d(x_{2k}, x_{2k}) + d(x_{2k+1}, x_{2k-1})]\})$

$\leq \varphi\left(\max\left\{sd(x_{2k+1}, x_{2k+1}), sd(x_{2k-1}, x_{2k}), \frac{1}{2}[sd(x_{2k-1}, x_{2k}) + sd(x_{2k}, x_{2k+1})]\right\}\right)$

$\leq \varphi(\max\{sd(x_{2k}, x_{2k+1}), sd(x_{2k-1}, x_{2k})\})$

Using Remark 1.9, if $sd(x_{2k+1}, x_{2k}) \leq \varphi(sd(x_{2k}, x_{2k+1}) < sd(x_{2k}, x_{2k+1})$, a contracdiction.

Thus, $sd(x_n, x_{n+1}) \leq \varphi(sd(x_{n-1}, x_n))$ for all $n = 2k,\ k \in N$. (3)

If $n = 2k + 1$, with a similar approach as above, we get

$$sd(x_n, x_{n+1}) \leq \varphi(sd(x_{n-1}, x_n))$$

for all $n = 2k + 1,\ k \in N$. (4)

(3) and (4) implies $sd(x_n, x_{n+1}) \leq \varphi(sd(x_{n-1}, x_n))$ for all $n \in N$.

By Induction, it can be seen that $sd(x_n, x_{n+1}) \leq \varphi^n(sd(x_0, x_1))$ for all $n \in N$.

Taking $Limitn \to \infty$ and using 1.8(ii), we get $\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta$. (5)

Applying Remark 1.9 to (2), we have $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in N$. (6)

Step 2: To prove that $\{x_n\}$ is a Cauchy sequence.

Let $\theta \ll c$ be given. Then there is a natural number $n_1 \in N$ such that

$d(x_n, x_{n-1}) \ll \frac{1-L(1+s)}{2s} c$, for all $n \geq n_1 \in N$. (7)

Let $m, n \in N$ with $m > n \geq n_1$.

We prove $d(x_n, x_m) \ll c$ for all $m > n \geq n_1$ by induction on m. (8)

Let $n \geq n_1$and $m = n + 1$. Using (2) and (7),

$d(x_n, x_m) = d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \ll \frac{1-L(1+s)}{2s} c \ll c$proving for $m = n + 1 \in N$.

Let (8) is true for $m \geq n + 1$. We have to prove for $m + 1$.

There are four possible cases.

Case (i): When $nisodd, m + 1\ iseven$.

$sd(x_n, x_{m+1}) = sd(fx_{n-1}, gx_m)$

$\leq \varphi(\max\{sd(x_{n-1}, x_n), sd(x_m, x_{m+1}), L[d(x_{n-1}, x_{m+1}) + d(x_n, x_m)]\})$

$\leq \varphi(\max\{sd(x_{n-1}, x_n), L[d(x_{n-1}, x_{m+1}) + d(x_n, x_m)]\})$ using (2)

$< \max\{sd(x_{n-1}, x_n), L[d(x_{n-1}, x_{m+1}) + d(x_n, x_m)]\}$ using Remark 1.9

If $sd(x_n, x_{m+1}) < sd(x_{n-1}, x_n)$, then using (5), $d(x_n, x_{m+1}) \ll c$.

If $sd(x_n, x_{m+1}) < L[d(x_{n-1}, x_{m+1}) + d(x_n, x_m)]$, then

$sd(x_n, x_{m+1}) < L[d(x_{n-1}, x_{m+1}) + d(x_n, x_m)]$

$\leq L[sd(x_{n-1}, x_n) + sd(x_n, x_{m+1}) + d(x_n, x_m)]$

$$i.e.,\ \ d(x_n, x_{m+1}) < \frac{L}{1-L}\left[d(x_{n-1}, x_n) + \frac{1}{s} d(x_n, x_m)\right]$$

$$< 2L\left[d(x_{n-1}, x_n) + \frac{1}{s} d(x_n, x_m)\right]$$

Using (7) and since (8) is true for m by induction hypothesis, we have

$d(x_n, x_{m+1}) \ll 2L\frac{1-L(1+s)}{2s} c + \frac{2L}{s} c \leq \frac{1-2L-L(s-1)}{s} c + \frac{2L}{s} c \leq \frac{1-2L}{s} c + \frac{2L}{s} c = \frac{1}{s} c \ll c$.

Thus (8) is true for $m + 1 \in N$. Therefore by Induction, (8) is true for all $m > n$ in this case.

Case (ii): When $niseven, m + 1\ isodd$. The proof of this is similar to the proof in case(i).

Case(iii): When $niseven, m + 1\ iseven$. From triangle inequality,

$d(x_n, x_{m+1}) \leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{m+1})$

$= sd(x_n, x_{n+1}) + sd(fx_n, gx_m)$

$\leq sd(x_n, x_{n+1}) + \varphi(\max\{sd(x_n, x_{n+1}), sd(x_m, x_{m+1}), L[d(x_n, x_{m+1}) + d(x_{n+1}, x_m)$

$=$
$sd(x_n, x_{n+1}) + \varphi(\max\{sd(x_n, x_{n+1}), L[d(x_n, x_{m+1}) + d(x_{n+1}, x_m)\})$, using (6)
$<$
$sd(x_n, x_{n+1}) + \max\{sd(x_n, x_{n+1}), L[d(x_n, x_{m+1}) + d(x_{n+1}, x_m)\}$, using Remark 1.9
If $d(x_n, x_{m+1}) \leq sd(x_n, x_{n+1}) + sd(x_n, x_{n+1}) = 2sd(x_n, x_{n+1})$ then using (6) and (7),
$d(x_n, x_{m+1}) < 2sd(x_{n-1}, x_n) \ll 2s\frac{1-L(1+s)}{2s}c \ll c$.
If $d(x_n, x_{m+1}) \leq sd(x_n, x_{n+1}) + L[d(x_n, x_{m+1}) + d(x_{n+1}, x_m)]$ then,
$$d(x_n, x_{m+1}) \leq sd(x_n, x_{n+1}) + L[d(x_n, x_{m+1}) + sd(x_{n+1}, x_n) + sd(x_n, x_m)]$$
$i.e.,$ $$(1-L)d(x_n, x_{m+1}) < (1+L)sd(x_n, x_{n+1}) + Lsd(x_n, x_m)$$
Using (6), (7) and the induction hypothesis, we have
$(1-L)d(x_n, x_{m+1}) \ll (1+L)s\frac{1-L(1+s)}{2s}c + Lsc < [1-L(1+s)]c + Lsc = (1-L)c$.
Thus $d(x_n, x_{m+1}) \ll c$ proving (8) for all $m > n$ in this case.
Case (iv): When $n is odd, m+1$ $is odd$. The proof of this is similar to the proof in case(iii).
Therefore,$\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, $\{x_n\}$ converges to some $p \in X$.
Step 3: To prove that $p \in X$ is the common fixed point of $f and g$.
Since $x_n \to p$, we have $x_{2n} \to p and x_{2n+1} \to p$.
Suppose f is continuous. Then $x_{2n} \to p$ imlies $x_{2n+1} = fx_{2n} \to fp$.
By uniqueness of the limit, we have $fp = p$. (9)
Now,
$sd(p, gp) = sd(fp, gp) \leq$
$\varphi(\max\{sd(p, fp), sd(p, gp), L[d(p, gp) + d(fp, p)]\})$
$= \varphi(sd(p, gp))$
If $\theta < sd(p, gp)$, then by Remark1.9, a contradiction.
Thus $d(p, gp) = \theta$or $gp = p$. (10)
From (9) and (10), $fp = gp = p$ $i.e.,$ p is the common fixed point of f and g.
Step 4: To prove that p is the unique common fixed point of f and g.
Let p and q be two distinct common fixed points of f and g. Then,
$sd(p, q) = sd(fp, gq)$
$\leq \varphi(\max\{sd(p, fp), sd(q, gq), L[d(p, gq) + d(fp, q)]\})$
$= \varphi(2Ld(p, q))$
Since $2L < 1 \leq s$, $sd(p, q) \leq \varphi(sd(p, q))$
If $\theta < d(p, q)$, then by Remark 1.9, we get a contradiction.
Thus $d(p, q) = \theta$ which implies $p = q$, proving the uniqueness.

Corollary 2.2 Let (X, d) be a complete cone b-metric space and with a real constant $s \geq 1$. f be a self-map on X. Suppose there is a constant $L < \frac{1}{1+s}$ and a comparision function φ over the cone $P \subseteq E$ satisfying the inequality
$sd(fx, fy) \leq \varphi(\max\{sd(x, fx), sd(y, fy), L[d(x, fy) + d(gx, y)]\})$for all $x, y \in X$
If f is continuous, then f has a unique fixed point.

Example 2.3 Let $E = R$, $P = [0, \infty)$and $X = [0,1]$. Define the metric as $d(x, y) = |x - y|^2$. Then (X, d) is a complete cone b-metric space with $s \geq 2$ Let us take $s = 3$. Consider the functions $\varphi: P \to P$, f and g as $\varphi(t) = \frac{t}{1+t}$; $f(x) = \frac{1}{3}x$; $g(x) = \frac{1}{12}x$ for all $x \in X$, $t \in P$. Then, here both f, g are continuous w.r.t d. The possible cases are $y = 4x$, $y > 4x$, $y < 4x$.
If $y = 4x$, then LHS of (1) is θ which implies the contractive condition (1) is satisfied.
If $y > 4x$, then $sd(fx, gy) = 3\left|\frac{y-4x}{12}\right|^2 \leq \frac{y^2}{48} \leq \frac{121y^2}{148+121y^2} = \varphi\left(\frac{121y^2}{48}\right) = \varphi(3d(y, gy))$
$i.e.,$ LHS of (1) ≤ RHS of (1) which implies the contractive condition(1) is satisfied.
If $y < 4x$,then $sd(fx, gy) = 3\left|\frac{4x-y}{12}\right|^{2\leq} \leq \frac{x^2}{3} \leq \frac{4x^2}{4x^2+3} = \varphi\left(\frac{4x^2}{3}\right) = \varphi(3d(x, fx))$.
Therefore, (1) is satisfied in all the three cases. By Theorem 2.1, f and g have unique common fixed point which is 0.

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