EQUIVALENCE BETWEEN CATEGORY OF BOOLEAN RINGS, BOOLEAN ALGEBRAS, A*-ALGEBRAS AND 3-RINGS

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Abstract: This paper presents definition of 3-ring, examples of 3-rings, definition of direct system of 3-rings, target for a direct system of 3-rings, direct limit for the direct system of 3-rings and theorems on direct system of 3-rings.

Keywords: 3-ring, direct limit, direct system and target.

Definition: A commutative ring (R, +, ., 1) such that $x^3 = x$, $3x = 0$ for all x in R is called a 3-ring.

Note: (1) x + x = -x for all x in a 3-ring R.
(2) Hereafter R-stands for a 3-ring.

Theorem: Suppose (R, +, ., 1) is a 3-ring. Then $B(R) = \{e \in R \mid e^2 = e\}$.Then B(R) is a Boolean algebra with $\wedge, \vee, (-)'$ defined by $e \wedge f = ef$, $e \vee f = (e + f) - ef$, $e' = 1 - e$ for all e, f∈B(R).

Theorem: (i) The Boolean algebra B(R) of idempotents of a 3-ring B(R) is a Boolean ring with ⊕,⊙ defined by $e \oplus f = e + f + ef$, $e \odot f = ef$ for all e, f ∈ B(R). (ii) For every a∈R, $e_a = 2a - a^2$, $f_a = a - a^2$. And clearly $e_a.f_a = 0$; Then clearly $B(R) = \{e_a, f_a \mid a \in R\}$.

Theorem: Every element a in R has a unique decomposition (normal form of a) given by $a = e_a - f_a$.

Proof: Let a ∈ R

$e_a - f_a = (2a - a^2) - (a - a^2) = a$

$e_a^2 = (2a - a^2)^2 = 4a^2 + a^4 - 2.2a^3$

$= a^2 + a^2 - 4a = 2a^2 - a = 2a - a^2$.

$\therefore e_a^2 = e_a$.

$(f_a)^2 = (a - a^2)^2 = a^2 + a^4 - 2a^3 = 2a^2 - 2a = a - a^2 = f_a$.

$\therefore (f_a)^2 = f_a$.

$e_a.f_a = (2a - a^2)(a - a^2) = 2a^2 - 3a^3 + a^2 = 3a^2 - 3a^3$

$= 0 - 0 = 0$.

$\therefore e_a.f_a = 0 \ \forall a \in R$.

Suppose a = e − f with ef = 0.

$a^2 = e^2 + f^2 - 2ef = e + f$

$\therefore a^2 = e + f$.

$2a - a^2 = 2(e - f) - (e + f) = 2e - 2f - e - f = e$.

$a - a^2 = (e - f) - (e + f) = -2f = f$

$\therefore a - a^2 = f$.

$\therefore e_a = e, f_a = f$.

∴ a has a unique normal decomposition.

Note: Suppose a, b ∈ R where R is a 3-ring. Then $a = e_a - f_a$ where $e_a = 2a - a^2$, $f_a = a - a^2$. Then

(i) $e_{a+b} = e_a + e_b + e_ae_b + f_af_b - e_af_b - e_bf_a$

(ii) $f_{a+b} = f_a + f_b + e_ae_b + f_af_b - e_af_b - e_bf_a$

(iii) $e_{ab} = e_ae_b + f_af_b$

(iv) $f_{ab} = e_af_b + e_bf_a$.

Theorem: The following are equivalent.

(a) The category of Boolean rings.
(b) The category of Boolean algebras.
(c) The category of A* - algebras.
(d) The category of 3-rings.

Proof: (a) ⇒ (b):

Suppose (R, +, ., 0, 1) is a Boolean ring. Define

$a \wedge b = ab$

$a \vee b = a + b - ab$

$a' = 1 - a$ for all a ∈ B(R).

Then (R, ∧, ∨, (−)′, 0, 1) is a Boolean algebra.

∴ (a) ⇒ (b).

(b) ⇒ (c):

Suppose (B, ∧, ∨, (−)′, 0, 1) is a Boolean algebra .

$A(B)=\{(a,b) \mid a,b \in B, a \wedge b = 0\}$. Then A(B) is an A* - algebra where

$\wedge, \vee, (-)^{\sim}, (-)_{\pi}, *, 1, 0$ are defined by

1) $(a_1, a_2) \wedge (b_1, b_2) = (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$ where juxta-position, +, (−)′ respectively ∧, ∨, (−)′ in Boolean algebra B.
2) $(a_1, a_2) \vee (b_1, b_2) = (a_1b_1 + a_1b_2 + a_2b_1, a_2b_2)$
3) $(a_1, a_2)^{\sim} = (a_2, a_1)$
4) $(a_1, a_2)_{\pi} = (a_1, a_1')$
5) $(a_1, a_2) * (b_1, b_2) = (a_1, a_1'b_2)$

∴ (b) ⇒ (c).

(c) ⇒ (d):

Suppose $(A, \wedge, \vee, *, (-)^{\sim}, (-)_{\pi}, 0, 1)$ is an A* - algebra.

Then (A, +, ., 1) is a 3-ring where +, . are defined as follows:

For a, b ∈ A,

$a + b = (a+b)_{\pi} * (a+b)^{\#}$

$ab = (ab)_{\pi} * (ab)^{\#}$ where

$(a+b)_{\pi} = (a^{\sim} \wedge b)_{\pi} \vee (a \wedge b^{\sim})_{\pi} \vee (a^{\#} \wedge b^{\#})$

$(a+b)^{\#} = (a \wedge b)_{\pi} \vee (a^{\#} \wedge b^{\sim}{}_{\pi}) \vee (a^{\sim}{}_{\pi} \wedge b^{\#})$

$(ab)_{\pi} = (a \wedge b)_{\pi} \vee (a^{\#} \wedge b^{\#})$

$(ab)^{\#} = (a_{\pi} \wedge b^{\#}) \vee (a^{\#} \wedge b_{\pi})$

∴ (c) ⇒ (d).

(d) ⇒ (a):

Suppose (R, +, ., 1) is a 3-ring.

$B(R) = \{e \in R \mid e^2 = e\}$. Then (B(R), ⊕, ⊙, 1) is a Boolean algebra where

$e \oplus f = e + f + ef$, $e \odot f = ef$.

∴ (d) ⇒ (a).

Theorem: Suppose $(B, \wedge, \vee, (-)', 0, 1)$ is a Boolean ring. $R(B) = \{(e,f) \mid e, f \in B, ef = 0\}$.
Then $(R(B), +, ., 0, 1)$ is a 3-ring where $+, ., 0, 1$ are defined by

(i) $(e_1, f_1) + (e_2, f_2) = (e_1 + e_2 + e_1e_2 + f_1f_2 - e_1f_2 - e_2f_1,\ f_1 + f_2 + e_1e_2 + f_1f_2 - e_1f_2 - e_2f_1)$

(ii) $(e_1, f_1)(e_2, f_2) = (e_1e_2 + f_1f_2, e_1f_2 + e_2f_1)$

(iii) $1 = (1,0)$, $0 = (0,1)$.

Example: $\mathbf{3} = \{0, 1, 2\}$. Then $(\mathbf{3}, +, ., 1)$ is a 3-ring where

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Theorem: Suppose X is a non empty set. Then $(\mathbf{3}^X, +, ., 0, 1)$ is a 3-ring where

(i) $(f+g)(x) = f(x) + g(x)$

(ii) $(f.g)(x) = f(x).g(x)$

(iii) $0(x) = 0$

(iv) $1(x) = 1$ for all $x \in X$, $f, g \in \mathbf{3}^X$.

Theorem: Suppose $\{R_i \mid i \in I\}$ is a family of 3-rings. Define $R = \prod_{i \in I} R_i = \{a \mid a : I \to \bigcup_{i \in I} R_i, a(i) \in R_i\}$.
Define $+, \cdot, -, 0, 1$ on R as follows:

(i) $(a+b)(i) = a(i) + b(i)$

(ii) $(a.b)(i) = a(i) . b(i)$

(iii) $(-a)(i) = -(a(i))$

(iv) $0(i) = 0$

(v) $1(i) = 1\ \forall\ i \forall I$ and $a,b \in R$.

Then R is a 3-ring.

References:

1. Simmons, G.F: Introduction to Topology and Modern Analysis, Mc Graw Hill Book Company Inc.
2. Vijaya Kumar, B: A*-algebras and 3-Rings, Ph.D. Thesis, Nagarjuna University, 2009, A.P., India.
3. Irving Kaplansky: Fields and Rings, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, 1972.
4. McCoy, N.H and Montgomery: A Representation of Generalised Boolean Rings, Duke Math.J., Vol.3(1937).
5. Serge Lang Addison: Algebra, Wesley Publishing Company, 1977.

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