GENERALIZATION OF RIEMANN INTEGRAL BASED ON GENERALIZED g-OPERATIONS

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Abstract: Ivana Stajner-Papuga, Tatjana Grbic and Martina Dankova introduced pseudo-Riemann Stieltjes integral in 2006. In this paper, we make a study of the generalization of classical Riemann integral in the pseudo-analysis framework. For that, we use the generalized g-operations, that is, generalized generated pseudo-addition ⊕ and the generalized generated pseudo-multiplication⊙. Let g be a positive strictly monotone function defined on [a, b], where [a, b] is a closed subinterval of [-∞, +∞], such that 0 ∈ Ran (g). The generalized generated pseudo-addition ⊕ and the generalized generated pseudo-multiplication ⊙ are given by $x \oplus y = g^{(-1)}(g(x) + g(y))$ and $x \odot y = g^{(-1)}(g(x)\ g(y))$, where $g^{(-1)}$ is pseudo-inverse function for function g. Using the generalized g-operations first we define Riemann pseudo - sum ⊕(**𝓟, f**) of the function f (where f: [c, d] → [a, b]) on [c, d] for the tagged partition **𝓟** and then the generalization of Riemann integral $(pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx$ of the function f.

Keywords: generalized generated pseudo-operations, g-semiring, pseudo-Riemann integral, Riemann pseudo-sum.

Introduction: Pseudo-analysis is the generalization of the classical analysis, where instead of the field of real numbers a semiring is defined on a real interval [a, b]⊆ [-∞, +∞] with pseudo-addition ⊕ and with pseudo-multiplication ⊙ *[2]*.Pseudo-analysis has applications in different fields, e.g., measure theory, integration, integral operators, convolution, Laplace transform, optimization, nonlinear differential and difference equations, economics, game theory, etc. *[7]*. Important fact is that this approach gives also solutions in the form that are not achieved by other theories *[2]*. In this paper, we make a study of the generalization of classical Riemann integral in the pseudo-analysis framework. First section contains some preliminary notions and the second section; we construct pseudo-Riemann integral and we got some results of that integral.

1 Preliminary Notions:

The basic preliminary notions needed in this paper are notion of generalized g-semiring. Before that, we give a short overview of the g-semiring.

Definition 1.1: ([1]-[5]) Let [a, b] be a closed subinterval of [-∞, +∞] (in some cases semi closed subintervals will be considered) and let ≤ be a total order on [a, b]. The operation⊕ is called a ***pseudo-addition*** if it is a function⊕: [a, b] x [a, b] → [a, b] which satisfies the following axioms: associativity, non-decreasing, a left neutral element or zero element **0**; that is **0**⊕x = x, for all x ∈ [a, b] and commutativity. The operation ⊙ is called a ***pseudo-multiplication*** (*[1]-[5]*) if it is a function ⊙: [a, b] x [a, b] → [a, b] which satisfies the following conditions: associativity, positively non-decreasing: that is, if x ≤ y implies x ⊙z≤ y ⊙ z, where $z \in [a,b]_+$ and $[a, b]_+ = \{x/x \in [a, b], \mathbf{0} \le x\}$, **1** is unit element: that is **1**⊙ x = x, for all x ∈ [a, b] and commutativity.

A ***semiring*** *([1],[3],[4],[5])*is the srtructure ([a, b],⊕, ⊙) where the following holds:

⊕ is pseudo addition;

⊙is psuedo - multiplication; **0**⊙ x = **0** and

x ⊙ (y ⊕ z) = (x ⊙ y) ⊕ (x ⊙ z), that is ⊙ is a distributive pseudo-multiplication with respect to ⊕.

Definition 1.2: ([1],[4],[5],[6]) **g-semiring** is the semiring with strict pseudo-operations defined by strictly monotone and continuous generating function g: [a, b]→ [0, +∞]. In this case we will consider only strict pseudo-addition, that is, such that the function ⊕ is continuous and strictly increasing in (a, b) × (a, b) *[7]*. By Arzel's representation theorem *[7]* for each strict pseudo-addition ⊕ there exists a monotone function g (generator for ⊕) g: [a, b] → [0, ∞] such that either g(a) = 0 or g(b) = 0 and $x \oplus y = g^{-1}(g(x) + g(y))$. Using a generator g of strict pseudo-addition⊕, we can define pseudo-multiplication ⊙ such that $x \odot y = g^{-1}(g(x)\ g(y))$. If the zero element of the pseudo-addition is 'a', we will consider increasing generators. Then g (a) = 0 and g (b) = ∞. If the zero element of the pseudo-addition is b, we will consider decreasing generators. Then g (b) = 0 and g (a) = ∞.

⊙ is distributive with respect to ⊕. Also

g(**0**) = 0 and g(**1**) = 1. In addition, since structure of semiring has to be maintained, *[1]* it is necessary to accept the convention

0. (+∞) = 0 (follows from **0**⊙ x = **0** for all

x∈ [a, b]). If the generator g is increasing (respectively decreasing), then the operation ⊕ induces the usual order (respectively opposite to the usual order) on the interval [a, b] in the following way;

x ≤ y if and only if g(x) ≤ g(y). Additionally, x < y if and only if g(x) ≤ g(y) and

x ≠ y.

Here we discuss only about g-operations. That is operations given by strictly monotone and continuous generator g: [a, b]→ [0, +∞] in the following manner:

$x⊕y = g^{-1}(g(x) + g(y))$,

$x ⊙y = g^{-1}(g(x)\ g(y))$. Therefore, in order to define generalized generated g-operations, it is necessary to use the pseudo-inverse ([4],[5]) function of the generator g instead of the classical inverse function.

Definition 1.3: ([4],[5]) For ***non-decreasing function*** f: [a, b] → [c, d], where [a, b] and [c, d] are closed subintervals of extended real line [-∞, +∞], ***pseudo - inverse*** is

$f^{(-1)}(y)$ = sup {x ∈ [a, b] / f(x) < y}. If f is a ***non - increasing function***, its ***pseudo - inverse*** is $f^{(-1)}(y)$ = sup {x ∈ [a, b] / f(x) > y}.

Definition 1.4: ([3]-[5]) Let g be a positive strictly monotone function defined on [a, b], where [a, b] is a closed subinterval of

[-∞, +∞], such that 0 ∈ Ran (g). The ***generalized generated pseudo-addition***⊕ and the***generalized generated pseudo-multiplication***⊙ are given by

$x⊕y = g^{(-1)}(g(x) + g(y))$ and

$x ⊙ y=g^{(-1)}(g(x)\ g(y))$, where $g^{(-1)}$ is ***pseudo-inverse*** function for function g.

Basic properties ([4],[5]) of generalized generated pseudo-operations from definition 1.4.

1. **I**f g(x) + g(y), g (z) g(x),
g(z) g(y) ∈ Ran(g), ⊙is distributive over ⊕.
2. Neutral element for ⊕ is $g^{(-1)}(0)$.
3. If 1 ∈ Ran (g), the unit element for ⊙ is $g^{(-1)}(1)$.
4. $g^{(-1)}(0) ⊙ x = g^{(-1)}(0) = x ⊙g^{(-1)}(0)$ for all x ∈ [a, b].
5. ⊕ is non - decreasing function.
6. ⊙ is positively non-decreasing function.
7. In the general case, associativity does not hold for ⊕.
8. In the general case, cancellation law does not hold for ⊕.
9. ⊕ and ⊙ are commutative.

Notation 1.5: ([3]-[5]) Since ⊕ is not necessarily associative operation, further on the following notation has been used:

$\bigoplus_{i=1}^{n} x_i = (\ldots\ldots((x_1⊕x_2)⊕x_3)⊕\ldots\ldots)⊕x_n,$

where x_i ∈ [a, b], i ∈ {1, 2, 3 ….n}.

Definition 1.6: ([3]-[5]) By the means of generating function g it is possible to introduce a ***metric***. Let

d: [a, b] x [a, b] → [0, ∞] be a function defined by d (x, y) = | g(x) – g(y) |,

where x, y ∈ [a,b] and g is a generating function for ⊕.

2 Pseudo-Riemann Integral:

In this section, first, we give the examples of some basic properties of the generalized generated pseudo-operations. Then we construct pseudo-Riemann integral using the generalized generated pseudo-operations and the above-mentioned metric.

Example 2.1: The following example shows that in the general case, associativity and cancellation law need not hold for generalized generated pseudo addition.

Let g: [0, +∞] → [0, +∞] given by

$$g(x) = \begin{cases} \ln(x+1) & ;\ x \in [0, 3], \\ e^x & ;\ x \in (3, +\infty] \end{cases}$$

be a generating function for pseudo-addition ⊕.

Its pseudo-inverse function is

$$g^{(-1)}(x) = \begin{cases} e^x - 1 & ;\ x \in [0, \ln 4], \\ 3 & ;\ x \in (\ln 4, e^3], \\ \ln x & ;\ x \in (e^3, +\infty]. \end{cases}$$

The generating function g is not continuous at x = 3. However, the pseudo-inverse function is continuous and strictly increasing on Ran (g). Now, for this choice of generating function and corresponding pseudo-operation, we can show that $(\frac{3}{2}⊕\frac{3}{4}) ⊕ 4 = \ln(\ln 4 + e^4) \neq \ln(\ln\frac{35}{8} + e^4) = \frac{3}{2}⊕(\frac{3}{4}⊕ 4)$.

That is, in general case, associative does not hold for ⊕.

Also we can show that $\frac{5}{2}⊕\frac{1}{2} = 3 = \frac{5}{2}⊕\frac{1}{3}$. However, $\frac{1}{3} \neq \frac{1}{2}$.That is, in the general case, the cancellation law does not hold for ⊕.

Definition 2.2: Let f: [c, d] → [a, b] be a function defined on an interval [c, d].Let g be generating function from definition 1.4 defined on interval [a, b] and $\oplus$ and $\odot$ generalized-generated pseudo - operations given by definition 1.4. If $\mathcal{P} = \{(w_i, (x_{i-1}, x_i])\}$, where i varies from 1 to n , is a tagged partition of [c, d], that is

$c = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = d$ and

$w_i \in (x_{i-1}, x_i]$, the ***Riemann pseudo - sum of f on [c, d] for the tagged partition $\mathcal{P}$*** is

$\oplus(\mathcal{P}, f) = \oplus_{i=1}^{n} f(w_i) \odot g^{(-1)}(x_i - x_{i-1})$.

Definition 2.3: The function f: [c, d] → [a, b] is ***pseudo-Riemann integrable*** on [c, d] whenever there is a real number P ∈ [a, b] satisfying the following condition: for each

$\varepsilon > 0$ there exists $\delta > 0$ such that

$d(\oplus(\mathcal{P}, f), P) < \varepsilon$, for all tagged partitions $\mathcal{P}$ of [c, d] that fulfills

$\max\{x_i - x_{i-1} / 1 \leq i \leq n\} < \delta$. If the number P from the previous definition exists, it is uniquely determined and is called ***pseudo-Riemann integral*** of f on [c, d], and it will be denoted by $(pR)\int_{[c,d]}^{(\oplus,\odot)} f dx$.

For g(x) = x the previous definition will give the classical Riemann integral $(R)\int_c^d f dx$. That is f is Riemann integrable on [c, d].

Theorem 2.4: (a) If g: [a, b]⟶ [0,+∞] is strictly monotone bijection and if f is pseudo-Riemann integrable on [c, d] then gof is a Riemann integrable on [c, d] and $(pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx = g^{-1}((R)\int_c^d gof\,dx)$.

(b) Let g: [a, b] → [0, +∞] be either strictly increasing, right - continuous or strictly decreasing, left – continuous such that

+∞ ∈ Ran (g) and let f: [c, d] → [a, b] be a pseudo - Riemann integrable function on

[c, d] then

$g((pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx) \geq (R)\int_c^d gof\,dx$, if the integral on the right hand side exists.

(c) Let g: [a, b] → [0, +∞] be either strictly increasing, left - continuous or strictly decreasing, right - continuous and let

f: [c, d] → [a, b] be a pseudo - Riemann integrable function on [c, d] then $g((pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx) \leq (R)\int_c^d gof\,dx$, if the integral on the right hand side exists.

Proof: (a) If g: [a, b]⟶ [0,+∞] is strictly monotone bijection then g-semiring is obtained. Then the pseudo-inverse and classical inverse of the generating function g is same. So $gog^{-1}(x) = x$ for all x ∈[0,+∞]. Since f is pseudo-Riemann integrable on [c, d], then by definition there is a real number P ∈ [a,b] satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\oplus(\mathcal{P}, f), P) < \varepsilon$, for all tagged partitions $\mathcal{P}$ of [c, d] that fulfills

$\max\{x_i - x_{i-1} / 1 \leq i \leq n\} < \delta$.

We have,

$\oplus(\mathcal{P}, f) = \oplus_{i=1}^{n} f(w_i) \odot g^{-1}(x_i - x_{i-1})$

$= g^{-1}(g(f(w_i))(x_i - x_{i-1}))$

Now, $g^{-1}(g(f(w_1))(x_1 - x_0))$

$\oplus g^{-1}(g(f(w_2))(x_2 - x_1))$

$= g^{-1}(g(f(w_1))(x_1 - x_0) +$

$g(f(w_2))(x_2 - x_1))$

Proceeding like this we get, $\oplus(\mathcal{P}, f) = g^{-1}(\sum_{i=1}^{n}(gof)(wi)(x_i - x_{i-1}))$

$d(\oplus(\mathcal{P}, f), P) < \varepsilon \Longrightarrow |g(\oplus(\mathcal{P}, f)) - g(P)| < \varepsilon \Longrightarrow |\sum_{i=1}^{n}(gof)(wi)(x_i - x_{i-1}) - g(P)| < \varepsilon$

This gives us the Riemann-integrability for gof. Hence $g(P) = (R)\int_c^d (gof)\,dx$.

That is $(pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx = g^{-1}((R)\int_c^d gof\,dx)$.

(b) Let g: [a, b] → [0, +∞] be strictly increasing right continuous. Since g is strictly increasing, we can write $\alpha = g^{(-1)}(x) =$

$\sup\{y \in [a, b] / g(y) < x\}$. By supremum property $g(\theta) \geq x$ for all $\theta > \alpha$. Since g is right continuous $g(\alpha) \geq x$. That is $go\,g^{(-1)}(x) \geq x$. Similarly, we can prove if g is strictly decreasing left continuous. Also given that

f: [c, d] → [a, b] be a pseudo - Riemann integrable function on [c , d], then by the definition, there is a real number P ∈ [a, b] satisfying the following: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\oplus(\mathcal{P}, f), P) < \varepsilon$, for all tagged partitions $\mathcal{P}$ of [c, d] that fulfills $\max\{x_i - x_{i-1} / 1 \leq i \leq n\} < \delta$. Also given that gof is Riemann integrable function on

[c, d]. Then by the definition, there is a real number $A = (R)\int_c^d (gof)\,dx$ satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$|\sum_{i=1}^{n} gof(w_i)(x_i - x_{i-1}) - A| < \varepsilon$ for all tagged partitions $\mathcal{P} = \{(w_i, (x_{i-1}, x_i])\}$, where **i** varies from 1 to n , of [c, d] that fulfills $\max\{x_i - x_{i-1} / 1 \leq i \leq n\} < \delta$.

Now, using $gog^{(-1)}(x) \geq x$, we get

$\oplus(\mathcal{P}, f) \geq g^{(-1)}(\sum_{i=1}^{n} gof(w_i)(x_i - x_{i-1}))$.

Hence

$g(\oplus(\mathcal{P}, f)) \geq \sum_{i=1}^{n} gof(w_i)(x_i - x_{i-1})$ ....(1)

$d(\oplus(\mathcal{P}, f), P) < \varepsilon \Longrightarrow |g(\oplus(\mathcal{P}, f)) - g(P)| < \varepsilon$.

$\Longrightarrow g(\oplus(\mathcal{P}, f)) - \varepsilon < g(P)$ .... (2)

$|\sum_{i=1}^{n} gof(w_i)(x_i - x_{i-1}) - (R)\int_c^d gof\,dx| < \varepsilon \Longrightarrow -\sum_{i=1}^{n} gof(w_i)(x_i - x_{i-1}) - \varepsilon <$

$-(R)\int_c^d gof\,dx$ ....(3)

From (1), (2) and (3),

$g\left((pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx\right) - (R)\int_c^d gof\,dx$

$> g(\oplus(\mathcal{P}, f)) - \varepsilon - \sum_{i=1}^{n} gof(w_i)(x_i - x_{i-1}) - \varepsilon$

$\geq -2\varepsilon$.

This holds for all $\varepsilon > 0$ and, after allowing $\varepsilon \to 0$, then we get

$g\left((pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx\right) \geq (R)\int_c^d gof\,dx$.

Similarly, we can prove (c) using $g \circ g^{(-1)}(x) \leq x$, if g: [a, b] → [0, +∞] is either strictly increasing left - continuous or strictly decreasing right – continuous.

Theorem 2.5: If f is Riemann integrable on [c, d] then g^{-1}o f is a pseudo - Riemann integrable function on [c, d], where g :[a, b]⟶ [0,+∞] is a strictly monotone bijection.

Proof: If f is Riemann integrable on [c, d], then by definition, there exists a number A having the following property: for every $\varepsilon > 0$, there exists a partition $\mathcal{P}\varepsilon$ of [c, d] such that for every partition $\mathcal{P}$ finer than $\mathcal{P}\varepsilon$ and for every choice of the points w_i in $(x_{i-1}, x_i]$, we have $\left| \sum_{i=1}^{n} f(w_i)(x_i - x_{i-1}) - A \right| < \varepsilon$. Since g is a strictly monotone bijection, g-semiring is obtained and hence $gog^{-1}(x) = x$ for all x in [0, +∞].

Now $\oplus(\mathcal{P}, g^{-1} o f) = g^{-1}\left(\sum_{i=1}^{n} f(w_i)(x_i - x_{i-1})\right)$. Hence, $\left|\sum_{i=1}^{n} f(w_i)(x_i - x_{i-1}) - A\right| < \varepsilon$ implies $\left| g(\oplus(\mathcal{P}, g^{-1} o f)) - A\right| < \varepsilon$. That means g^{-1}o f is a pseudo - Riemann integrable function on [c, d] and

$g\left((pR)\int_{[c,d]}^{(\oplus,\odot)} g^{-1} o f\,dx\right) = A = (R)\int_c^d f\,dx$.

Theorem 2.6: Let g: [a, b] →[0, +∞] be strictly monotone bijection. Then f: [c, d] → [a, b] is pseudo Riemann integrable on [c, d] if and only if for each $\varepsilon > 0$ there is a partition $\mathcal{P}_\varepsilon$ of [c, d] such that

$U(\mathcal{P}_\varepsilon, gof) - L(\mathcal{P}_\varepsilon, gof) < \varepsilon$.

Proof: Let g: [a, b] → [0, +∞] be strictly monotone bijection. First, assume that f: [c, d] → [a, b] is pseudo Riemann integrable on [c, d]. Then by theorem: 2.4, gof is Riemann-integrable on [c, d]. Then by Riemann Criterion for integrability [**8**], for each $\varepsilon > 0$ there is a partition $\mathcal{P}_\varepsilon$ of [c, d] such that $U(\mathcal{P}_\varepsilon, gof) - L(\mathcal{P}_\varepsilon, gof) < \varepsilon$.

Conversely assume that $U(\mathcal{P}_\varepsilon, gof) - L(\mathcal{P}_\varepsilon, gof) < \varepsilon$, for each $\varepsilon > 0$ and $\mathcal{P}_\varepsilon$ is a partition of [c, d].Then by Riemann Criterion for integrability gof is Riemann integrable on [c, d]. Since gof is Riemann integrable on [c, d] and g is a strictly monotone bijection then by theorem: 2.5, g^{-1}o (gof) is pseudo- Riemann integrable on [c, d]. That is f is pseudo-Riemann integrable on [c, d].

Theorem 2.7: Let f: [c, d] → [a, b] be pseudo-Riemann integrable on [c, d] and let g: [a, b] → [0, +∞] be strictly monotone bijection. Let F: [c, d] → [0, +∞] satisfy the conditions:

(a) F is continuous on [c, d]

(b) The derivative F' exists and $F'(x) = (gof)(x)$ for all x∈ (c, d).

Then $(pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx = g^{-1}(F(d) - F(c))$.

Proof: Given that f: [c, d] → [a, b] is pseudo Riemann integrable on [c, d] and g: [a, b] → [0, +∞] is a strictly monotone bijection. Then by theorem: 2.4, gof is Riemann-integrable on [c, d]. Also given that F: [c, d] → [0, +∞] is continuous on [c, d] and the derivative F' exists and $F'(x) = (gof)(x)$ for all x∈ (c, d). Then by first form of fundamental theorem of calculus ***[8]***, we can write $(R)\int_c^d (gof)\,dx = F(d) - F(c)$. We have $(pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx = g^{-1}\left((R)\int_c^d (gof)\,dx\right)$.

Hence $(pR)\int_{[c,d]}^{(\oplus,\odot)} f\,dx = g^{-1}(F(d) - F(c))$.

Theorem 2.8: Let f: [c, d] → [a, b] be pseudo Riemann integrable on [c, d] and let g:[a, b] → [0, +∞] be strictly monotone bijection. Let $F(x) = \int_c^x (gof)dx$ for x ∈ [c, d]; then F is continuous on [c, d]. Moreover if (gof) is continuous at a point a ∈ [c, d], then F is differentiable at 'a' and $F'(a) = (gof)(a)$.

Proof: Given that f: [c, d] → [a, b] is pseudo Riemann integrable on [c, d] and g:[a, b] → [0, +∞] is a strictly monotone bijection. Then by theorem: 2.4, gof is Riemann integrable on [c, d]. Also given that $F(x) = \int_c^x (gof)dx$ for x ∈ [c, d]. Then by second form of Fundamental theorem of calculus [**8**], F is continuous on [c, d]. Also if (gof) is continuous at a point a ∈ [c, d], then by the same theorem, F is differentiable at 'a' and $F'(a) = (gof)(a)$.

Conclusion: The main aim of this paper is to present pseudo-analysis counterpart of the classical Riemann integral. The first two theorems give the relation between pseudo-Riemann integral and classical Riemann integral in different cases.

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