ANALYSIS OF TRANSIENT BEHAVIOR M/M/1 QUEUEING MODEL WITH CATASTROPHICAL EVENTS

M.RENI SAGAYARAJ, S.ANAND GNANA SELVAM, P. MANOHARAN, R.REYNALD SUSAINATHAN

Abstract: In this paper, we illustrate how generating function technique can be used to obtain analysis the transient bahavior of M/M/1 queueing model with catastrophical events. We provide the steady state probabilities of the queueing model M/M/1 and certain performance measures are deduced. We introduce the cost connected with the busy period idle periods of the queueing system and present a random motion associated with the cost structure. An elaborate procedure is used to derive partial differential equations governing the joint probability density function associated with catastrophe. Numerical illustrations are provided to see the effect of parameters on system performance measures.

Keywords: M/M/1 queue, catastrophes, transient analysis, steady state analysis, busy period analysis.

Introduction: The queueing Theory has played an important role in the theory of probability and related concepts. Its applications have been utilized varies fields like communication system, industrial sector and so on. Human beings, telephone calls flow of finished products, failed machines and so on may be considered as queueing units. In modern days, the queueing models have been analyzed by assuming the telephone calls as the units for demanding service.

Markov process modulated queueing models play a significant role in the ATM networks (Elvalid and Mitra [6], Anick et. al. [2], Simonian and Virtomo [16]). In addition queueing models driven by finite state space Markov processes that modulate the input rate in the queueing buffer have been analyzed by many authors (Anick et al. [2], Coffman et al. [4], Gaver et al. [5], Mitra [6], Low and Varaiya [8]). Noam Paz and Yechiali, U [13] have proposed a spectral-decomposition method to analyze a queueing model driven by an M/M/1 queue. Adan and Resing [1] have analyzed and expressed the generalized eigen values explicitly using the Chebyshev polynomials of the second kind. Van Doorn and Scheinhardt [13] have obtained explicit expressions for the stationary distribution of the buffer content for queueing queues driven by an M/M/1 queue with constant arrival service rates. Much of the vast literature on queueing models is confined to results describing steady-state operation only. But in many potential application of queueing theory, the practitioner needs to know how the system will operate up to some instant t. Many systems begin operation and are stopped at some specified time t. Business or service operations such as rental agencies or physician's offices which open and close, never operate under steady-state conditions. Furthermore, if the system is empty initially, the fraction of time the server is busy and the initial rate of output, etc., will be below the steady-state values, and hence, the use of steady-state results to obtain these measures is not appropriate. Thus, the investigation of the transient behavior of queueing processes is also important from the point of view of the theory and its applications.

The rest of the paper is organized as follows. The mathematical description of the model is in Section 1 and the governing equations of the model are given in Section 2. A Random motion Associated with the Queueing models and the transient solution has been derived in Section 3. An M/M/1/∞ queue with catastrophical events and the corresponding steady state results have been obtained and analyzed when there are no catastrophes in Section 4.

1. Mathematical description of the Model Arrival: We assume that customers arrive according to a Poisson process with rate λ

Server: Single server M/M/1 queueing system and the capacity 1.

Catastrophes: The catastrophes occur at the server as an independent Poisson process with parameter and inactivate the server upon arrival.

This means that all customers who arrive to the system find no waiting room and are lost when the server is busy in serving a customer already there in the system. In other words the waiting room capacity is 0. The service time is random having exponential distribution with mean $\frac{1}{\mu}$. Let $P_n(t)$ be the probability that there are n customers in the system at time t. Since the waiting room capacity is 0. We note that.

$$P_0(t) + P_1(t) = 1 \quad (2.1)$$

And $P_n(t) = 0$ for all $n \geq 2$. we assume $P_0(0) = a$ and $P_1(0) = b$. Then a +b= 1.

2. Governing Equations.

$$P_0(t + \Delta) = P_0(t)[1 - \lambda\Delta] + P_1(t)\mu\Delta + 0(\Delta)$$
$$P_1(t + \Delta) = P_1(t)[1 - \mu\Delta] + P_0(t)\lambda\Delta + 0(\Delta)$$

From the above equations, we have

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \quad (2.2)$$
$$P_1'(t) = -\mu P_1(t) + \lambda P_0(t) \quad (2.3)$$

Setting $g_0(t) = P_0(t)e^{(\lambda+\mu)t}$,

$g_1(t) = P_1(t)e^{(\lambda+\mu)t}$

The above equations yield

$$g_0(t) = g_0(0) + \frac{\mu}{\lambda+\mu}\{e^{(\lambda+\mu)t} - 1\} \quad (2.4)$$

$$g_1(t) = g_1(0) + \frac{\lambda}{\lambda+\mu}\{e^{(\lambda+\mu)t} - 1\} \quad (2.5)$$

Since $g_0(t)$ = a and $g_1(t) = b$ the equations (2.4) and (2.5) yields

$$P_0(t) = \frac{\mu}{\lambda+\mu}\{1 - e^{-(\lambda+\mu)t}\} + ae^{-(\lambda+\mu)t} \quad (2.6)$$

$$P_1(t) = \frac{\lambda}{\lambda+\mu}\{1 - e^{-(\lambda+\mu)t}\} + be^{-(\lambda+\mu)t} \quad (2.7)$$

If we choose $a = \frac{\lambda}{\lambda+\mu}$ then $b = \frac{\lambda}{\lambda+\mu}$ and the equations (2.6) and (2.7) yields.

$$P_0(t) = a \& P_1(t) = b \text{for all l} \quad (2.8)$$

From equation (2.8) we observe that if the initial probabilities are assumed to be the steady state values, then the queueing process is stationary at any time. If we assume that the process starts with I customer at time t =0, then a = 0 and b = 1. In this situation, we have the transient state probabilities given by

$$P_0(t) - \frac{\mu}{\lambda+\mu}\{1 - e^{-(\lambda+\mu)t}\} \quad (2.9)$$

$$P_1(t) - \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t} \quad (2.10)$$

The distributions of the busy and idle periods of this queueing system with the above initial condition are easy to obtain. let $\phi(t)$ and $\psi(t)$ be respectively the probability density functions of the busy and idle periods. As a busy period is defined to be the time intervals between the time of arrival of a customer to an idle server and he epoch when the server next becomes idle, we note in this queueing model that

$\phi(t) = \mu e^{-\mu t}$ and $\psi(t) = \lambda e^{-\lambda t}$

The probability density function of the busy cycle is the convolution of the above two densities and it is given by

$$\phi(t) \odot \psi(t) = \frac{\lambda\mu}{\lambda+\mu}\{e^{-\mu t} - e^{-\lambda t}\}$$

Althrough the distributions of the busy and idle periods obtained in a simple way , they can also be derived by considering the state probabilities. To do this , we postulate that there is an absorbing barrier at zero system size. If $q_n(t)$ denote the state probabilities of this restricted system. Then the differential equations for the state probabilities are given by.

$$q_0'(t) = \mu q_1(t) and q_1'(t) = -\mu q_1(t) \quad (2.12)$$

Clearly $q_n(t) = 0$ for all n ≥ 0. Since $q_1(0) = 1$ we get from (2.12) that

$q_1(t) = e^{-\mu t}$ and $q_0(t) = 1 - e^{-\mu t}$

We note that $q_0(t)$ gives the distribution function of the busy period hence its probability density function is given by $\mu e^{-\mu t}$. In a similar manner the density of the idle period can be found as $\lambda e^{-\lambda t}$ by postulating that there is an absorbing barrier at state 1. In the next section we make use of the distributions of the busy and idle periods to investigate a random motion connected with queueing system.

3. A Random motion Associated with the Queueing models: We consider the queueing models M/M/1 starting with one customer at time t=0 for this model. We have already mentioned that the busy periods alternate on the time axis. as busy periods always fetch utility of the system by the customers and in turn provide profit to the system, we attach a positive cost $C_1, C_1 > 0$ per unit time to busy periods. In the same way, a negative cost $-C_2, C_2 > 0$ per unit time is attached to idle periods. Considering the epochs of beginning and ending of busy periods, we note that these epochs constitute an alternating point process on the time axis characterized by two densities $\mu e^{-\mu t}$ and $\lambda e^{-\lambda t}$. Let N(t) be the number of such epochs which have occurred in the interval (0 , t]. Let C(t) be the cost per unit time at time t. then it can be easily seen that

$$C(t) = \frac{1}{2}(C_1 + C_2) + \frac{1}{2}(C_1 + C_2)(-1)^{N(t)} \quad (3.1)$$

Let P(t) be the net gain of the system upto time t, Then it can be easily seen that.

$$P(t) = \int_0^t C(u)du \quad (3.2)$$

Substituting (3.1) and (3.2) we get

$$C(t) = \frac{1}{2}(C_1 - C_2)t + \frac{1}{2}(C_1 + C_2) + \int_0^t (-1)^{N(t)}du \quad (3.3)$$

The two random quantities P(t) and C(t) constitute a random motion and it is studied in the next section .

4. An M/M/1/∞ queue with catastrophical events. Let customers arrive at a single server queueing system according to a Poisson process with rate λ. We assume that the service time has exponential distribution with parameter μ. Let the service discipline be FIFO. We assume that the system capacity is infinite. Let the catastrophic events arrive independently at the service facility according to a Poisson process with rate γ. the nature of an catastrophic events is that upon its arrival at the service station , it destroys all the customers there waiting and in the service and also renders the server momentarily inactive that is becomes ready for service immediately.

Let $P_n(t)$ be the probability that there are n customers in the system at time t. Then by the routine procedure, we have

$$P_n'(t) = \mu P_{n+1}(t) - (\lambda + \mu + \gamma)P_n(t) + \lambda P_{n-1}(t), n = 1,2, \ldots \quad (4.1)$$

$$P_0'(t) = \mu P_1(t) - \lambda P_0(t) + \gamma[1 - P_0(t)],$$

Where λ and μ have the usual meanings. We assume that a customer arrives to an system at the service faculty at time t = 0 so that the busy period starts at time t =0.

Then $P_n(0) = \delta_{n,1}$, $n = 0,1,2 \ldots\ldots$ *Where*

$$\delta_{n,1} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

To solve (3.1) We proceed as follows Defining

$$P(s,t) = \sum_{n=0}^{\infty} P_n(t) s^n$$

We obtain from (4.1)

$$\frac{\partial P(s,t)}{\partial t} = \left[\lambda s + \frac{\mu}{s} - (\lambda+\mu+\gamma)\right] P(s,t) + \mu\left(1-\frac{1}{s}\right) P_0(t) + \gamma \qquad (4.2)$$

Subject to the condition $P(s,0) = s$. The equation (4.2) can be solved and we obtain $P(s,t) = se^{-(\lambda+\mu+\gamma)} e^{\left(\lambda s + \frac{\mu}{s}\right)t} + \mu\left(1-\frac{1}{s}\right)\int_0^t P_0(u)\, e^{-(\lambda+\mu+\gamma)(t-u)} e^{\left(\lambda s+\frac{\mu}{s}\right)(t-u)} du + \int_0^t e^{\left(\lambda s + \frac{\mu}{s}\right)(t-u)}\, du$ (4.3)

In (4.3) we use the generating function

$$exp\left\{\left(\frac{\mu}{s} + \mu s\right)t\right\} = exp\left\{\frac{1}{2}\left((\beta s) + \frac{1}{(\beta s)}\right)(\alpha t)\right\} = \sum_{-\infty}^{\infty} (\beta s)^n I_n(\alpha t) \qquad (4.4)$$

Where we have set $\lambda = \frac{\alpha\beta}{2} and \mu = \frac{\alpha}{2\beta}$ we have $\alpha = 2\sqrt[2]{\lambda\mu}$ and $= \sqrt{\frac{\lambda}{\mu}}$. In the above $I_n(\alpha t), n = 0, \pm 1, \pm 2, \ldots\ldots$ are modified bessel functions of the first kind given by

$$I_n(u) = \sum_{k=0}^{\infty} \frac{u^{n+2k}}{2^{n+2k} k!\,(n+k)!}\ , n > -1\ ;\ I_{-n}(u) = I_n(u)$$

Then the equating the power of s^n on the both sides, we obtain as in previous section

$$P_0(t) = \frac{1}{\mu}\sum_{n=1}^{\infty} \frac{(n+1)}{\beta^{n+1}} \frac{I_{n+1}(\alpha t)}{t} e^{-(\lambda+\mu+\gamma)} + \frac{\gamma}{\mu}\int_0^t \frac{n}{\beta^n} \frac{I_n(\alpha u)}{u} e^{-(\lambda+\mu+\gamma)t} du$$

$$P_n(t) = \frac{2\gamma\beta^{n+1}}{\alpha}\int_0^t \sum_{k=0}^{\infty} \frac{(n+k+1) I_{n+k+1}(\alpha u)}{\beta^{k+1} u} e^{-(\lambda+\mu+\gamma)u} du + \sum_{m=0}^{\infty} e^{-(\lambda+\mu+\gamma)t}\left[\frac{I_{m+n+2(\alpha t)}}{\beta^{m-n+2}} + \frac{I_{m+n+3(\alpha t)}}{\beta^{m-n+1}}\right] + \beta^{n-1} I_{n-1}(\alpha t) e^{-(\lambda+\mu+\gamma)t}\ , n = 1,2 \ldots$$

The above probabilities completely describe the queueing process.

Conclusion: In this paper, we considered a server queue in which the inter arrival times follow exponential distribution and service times follow Poisson distribution has been considered. When service is going on catastrophe occurs.For this model, through probability generating functions and the expected queue size, under the system is down as well as that is functioning have been derived.

References:

1. I.J.B.F. Adan and J.A.C.Resing, Simple analysis of a fluid queue driven by an M/M/1 queue, Queueing Systems, 22 (1996), 171-174.
2. A.Devi, A. Srinivasan, Expected Time to Recruitment for A Single Grade Manpower; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 887-890
3. D. Anick, D. Mitra and M.M. Sondhi, Stochastic theory of a data-handling system with multiple sources, Bell Syst. Tech. J, 61 (1982), 1871-1894.
4. Ayyappan. G, Devipriya.G, (2013) - "Transient Analysis of Single server queueing system with Batch service under catastrophe". International Journal of Mathematical Archive. Vol.4(5), 26-32.
5. W.G. Coffman, B.M. Igelnik and Y.A. Kogman, Controlled stochastic model of a communication system with multiple sources, IEEE Trans.Inform. Theory, 37 (1991), 1379-1387.
6. X. Chao, A queueing network model with catastrophes and product form solution, O.R. Letters 18, 75-79,(1995),
7. A. Elvalid and D. Mitra, Effective bandwidth of general Markovian sources and admission control of high-speed networks, IEEE/ACM Transactions on Networking, 1 (1993), 329-343.
8. D.P. Gaver and J.P. Lehoczky, Channels that cooperatively service a data stream and voice messages, IEEE Trans. Commun., 30 (1982), 1153-1162.
9. N.K. Jain and Rakesh Kumar, A catastrophic queueing model with correlated input for the cell traffic generated by new broadband services, Indian J. of Mathematics and Math. Sciences, 1, No 2 (2005), 99-108.
10. N.K. Jain and Rakesh Kumar, Transient solution of catastrophic-cum-restorative queueing problem with correlated arrivals and variable service capacity, Information and Management Sciences, 18 (2007), 461-465.
11. A.Praveenprakash,Beena James,Joshi Nambikkai Rani, Study of the Different Types of Curriculum, Related; Mathematical Sciences

International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 307-311

12. B. Krishna Kumar, P.R. Parthasarthy and M. Sharafali, transient solution of an M/M/1 queue with balking, Queueing Systems, (Theory and Applications) 13, 441-447, (1993).
13. B. Krishna Kumar, D. Arivudainambi, Transient Solution of an M/M/1 Queue with Catastrophes, Computers & Mathematics with applications,40 (2000), 1233-1240.
14. J. Medhi, Stochastic Models in Queueing Theory, California: Academic Press,(2006).
15. Noam Paz and Yechiali, U, (2014) - "An M/M/1 queue in Random Environment with Disasters Asia – Pacific Journal of Operational Research. Vol. 31, 3, 1450016.
16. A.Praveenprakash, N.Lakshmipathy, P.Anitha, Problems of Housemaids in Chennai City – A Study; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 343-346
17. P.R. Parthasarathy, A transient solution to an M/M/1 queue: a simple approach, Adv. Appl. Prob., 19 (1987), 997-998.
18. Sudhesh, R (2010) - "Transient analysis of a queue with system disasters and customer impatience", queueing systems, Vol.66, 95-105.
19. J. Virtamo and I. Norros, Fluid queue driven by an M/M/1 queue, Queueing Systems, 16 (1994), 373-386.
20. M. Subhas Abel, Veenam.Basangouda, Magnetohydrodynamic Free Convection Heat Transfer; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 326-335

***

M.Reni Sagayaraj, S.Anand Gnana Selvam, P. Manoharan/
Department of Mathematics/ Sacred Heart College(Autonomous)/
Tirupattur – 635601/ Vellore District/ Tamil Nadu/ S.India.
R.Reynald Susainathan/Reckitt Benckiser/ India Limited (Gurgaon)/ New Delhi/ India.