ON 2-ABSORBING AND WEAKLY 2-ABSORBING IDEALS OF LATTICES

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Abstract: In this paper, we introduce 2-absorbing and weakly 2-absorbing ideals in lattices. We study their properties such as every 2-absorbing ideal of a lattice with zero is weakly 2-absorbing ideal. We define the triple zero in lattices and give some results related triple zero. Examples and counter examples are given wherever required.

Keywords: 2-absorbing ideal, prime ideal, weakly prime ideal, weakly 2-absorbing ideal in lattices.

Introduction: Ideals were first proposed by Richard Dedekind in 1876 in the third edition of his book Vorlesungen ber Zahlentheorie (English: Lectures on Number Theory). This was a generalization of the concept of ideal numbers developed by Emst Kummer. Later this concept was expanded by David Hilbert and especially Emmy Noether.

In 2003 Anderson and Smith [2], defined a weakly prime ideal in a commutative ring R, that is a proper ideal $P$of $R$with the property that, if whenever $a, b \in R$, $0 \neq ab \in P$implies either $a \in P$or $b \in P$.

Badawi [5] in 2007 defined a proper ideal $I$of a commutative ring $R$to be a 2-absorbing ideal, if whenever $abc \in I$for $a, b, c \in R$, then either $ab \in I$or $bc \in I$or $ac \in I$. Later this concept was generalized by Anderson and Badawi [3], Payroviand Babaei [12], Azizi [4], Badawi and Darani [6] and Chaudhari [9].

In this paper we introduce the concepts of 2-absorbing and weakly 2-absorbing ideals in lattices. A proper ideal $I$of a lattice $L$is called 2-absorbing if whenever $a \wedge b \wedge c \in I$ for $a, b, c \in L$, then either $a \wedge b \in I$or $a \wedge c \in I$or$b \wedge c \in I$. A proper ideal $I$of a lattice $L$with zero is called weakly 2-absorbing if whenever $0 \neq a \wedge b \wedge c \in I$ for $a, b, c \in L$, then either $a \wedge b \in I$ or $a \wedge c \in I$or $b \wedge c \in I$.

In Section 2, we study some basic properties of prime ideals, weakly prime ideals, 2-absorbing and weakly 2-absorbing ideals in lattices and give some examples. In Section 3, we study the concepts of a triple zero and derive some related results. Let Ibe a weakly 2-absorbing ideal of a lattice $L$with zero and $a, b, c \in L$. We say that (a, b, c) is a triple-zero of Iif $a \wedge b \wedge c = 0$, then $a \wedge b \notin I$, $a \wedge c \notin I$and $b \wedge c \notin I$.

We assume throughout that all lattices are lattices with zero.

2. Basic Properties of 2-absorbing and Weakly 2-absorbing Ideals in Lattices: We recall some concepts from the lattice theory, see Gratzer [10].

Definition 1: A set $P$with a binary relation '$\leq$' is called partially ordered set or poset if $\leq$is reflexive, transitive and antisymmetric.

Definition 2: A supremum (resp. infimum) is defined as follows. Let $H$is subset of a poset P, $a \in P$. Then $a$is an upper bound (resp. a lower bound) of H, if $h \leq a$(resp. $a \leq h$) for all $h \in H$. An upper bound (resp. a lower bound) a of H is the least upper bound (resp. the greatest lower bound) of Hor supremum (resp. infimum) of Hif, for any upper bound (resp. any lower bound) $b$of H, we have $a \leq b$(resp. $b \leq a$). We shall write $a = supH$(resp. $a = infH$), or $a = \vee H$ (resp. $a = \wedge H$).

Definition 3: Let $(L, \leq)$be a poset. Then $L$is called a lattice if for all $a, b \in L$, $sup\{a, b\}$and $inf\{a, b\}$exists.

Definition 4: A lattice Lhas a zero element, 0if $0 \leq x$, for all $x \in L$.

Definition 5: A sublattice $I$of $L$is an ideal if $i \in I$and $a \in L$imply that $a \wedge i \in I$.

Definition 6: A proper ideal $I$of a lattice $L$is called prime if $a, b \in L$and $a \wedge b \in I$imply that either $a \in I$or $b \in I$.

Example 1: Consider the lattice shown in Figure 1. Here the ideal $I = \{0, a, b, d\}$is prime ideal.

Definition 7: Let Lbe a lattice with zero. A proper ideal $I$of $L$is called weakly prime if $a, b \in L$and $0 \neq a \wedge b \in I$imply that either $a \in I$or $b \in I$.

Example 2: Consider the lattice shown in Figure 1. Here the ideal $I = \{0, b, c, f\}$is weakly prime ideal.

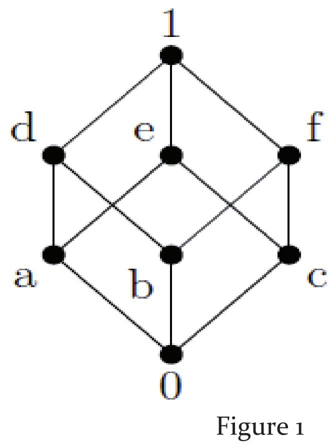

Figure 1

Definition 8: Let Lbe a lattice. A proper ideal $I$of $L$is called a 2-absorbing ideal if whenever$a \wedge b \wedge c \in I$ for $a, b, c \in L$, then either $a \wedge b \in I$or $a \wedge c \in I$or $b \wedge c \in I$.

Example 3: Consider the lattice shown in Figure 2. Here the ideal $I = \{0, b, c, f\}$is 2-absorbing ideal.

Definition 9: Let Lbe a lattice with zero. A proper ideal $I$of $L$is called a weakly 2-absorbing ideal if whenever $0 \neq a \wedge b \wedge c \in I$ for $a, b, c \in L$, then either $a \wedge b \in I$or $a \wedge c \in I$or $b \wedge c \in I$.

Example 4: Consider the lattice shown in Figure 2. Here the ideal $I = \{0, a, c, e\}$is weakly 2-absorbing ideal.

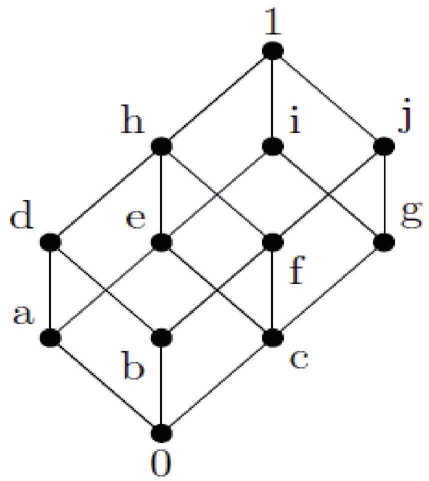

Figure 2

Lemma 2.1: Every prime ideal of a lattice $L$with zero is a weakly prime ideal.

Proof. Let Ibe a prime ideal of L. suppose that $a, b \in L$and $0 \neq a \wedge b \in I$. As $I$is prime ideal of L, we have either $a \in I$or $b \in I$. Thus $I$is a weakly prime ideal of L.

Remark 2.1: The following example shows that the converse of this Lemma does not hold. We give one counter example.

Example 5: Let L be a lattice shown in Figure 1. The ideal $I = \{0\}$is weakly prime ideal, but for $a \wedge b = 0 \in I$, we have neither $a \in I$nor $b \in I$. Thus $I$is not a prime ideal.

Lemma 2.2: Every prime ideal of a lattice $L$is a 2-absorbing ideal of L.

Proof. Let Ibe a prime ideal of L. Suppose that $a, b, c \in L$and $a \wedge b \wedge c \in I$. As $I$is prime ideal of L, we have either

(1) $a \wedge b \in I$or $c \in I$, or (2)$a \wedge c \in I$or $b \in I$,
or (3) $b \wedge c \in I$or $a \in I$.

Without lost of generality, suppose that $a \wedge b \in I$or $c \in I$. If $a \wedge b \in I$then the proof is obvious and if $c \in I$then $a \wedge c \in I$and $b \wedge c \in I$. Thus $I$is a 2-absorbing ideal of L.

Lemma 2.3: Every weakly prime ideal of a lattice $L$with zero is a weakly 2-absorbing ideal of L.

The proof of this Lemma is obvious.

Remark 2.2: The converse of the preceding is not true. We give a counter example.

Example 6: Consider the lattice shown in Figure 2. The ideal $I = \{0, b, c, f\}$is a 2-absorbing and a weakly 2-absorbing ideal.

For$h, j \in L$, $h \wedge j = f \in I$, neither $h \in I$nor $j \in I$. Hence $I$is neither prime norweakly prime ideal.

Lemma 2.4: Every 2-absorbing ideal of a lattice $L$with zero is a weakly 2-absorbing ideal of L.

Proof. Suppose that $I$is a 2-absorbing ideal of a lattice L. Let $a, b, c \in L$and $0 \neq a \wedge b \wedge c \in I$. As $I$is 2-absorbing ideal of L, we have either $a \wedge b \in I$or $a \wedge c \in I$ or $b \wedge c \in I$.

Remark 2.3: The converse of this Lemma does not hold. Here we give one counter example.

Example 7: Consider the lattice shown in Figure 1. Here the ideal $I = \{0\}$is a weakly 2-absorbing ideal. For $d, e, f \in L$, we have $d \wedge e \wedge f = 0 \in I$we have neither $d \wedge e = a \in I$nor $d \wedge f = b \in I$nor $e \wedge f = c \in I$. Thus $I$is not a2-absorbing ideal.

The following lattice contains an ideal that is neither 2-absorbingnor weakly 2-absorbing.

Example 8: Consider the lattice shown in Figure 3. Let $I = \{0, c\}$. Then $I$is the ideal of this lattice.

For $k, m, n \in L$such that $k \wedge m \wedge n = c \in I$, we have neither $k \wedge m = f \in I$nor $k \wedge n = h \in I$nor $m \wedge n = j \in I$.

Thus $I$is neither weakly 2-absorbing nor 2-absorbing ideal of L.

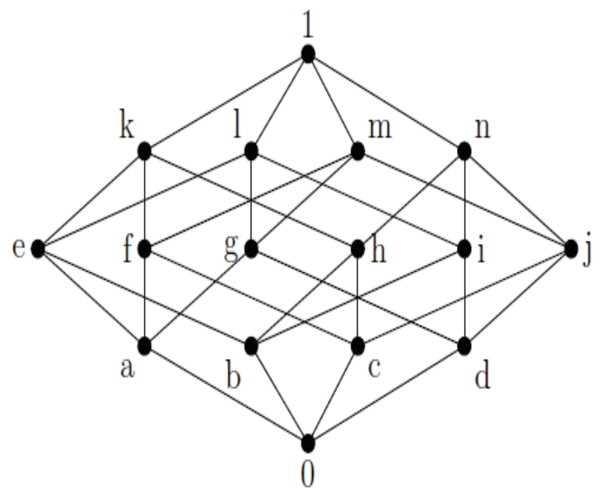

Figure 3

Lemma 2.5: Let $P$and Qbe two distinct prime ideals of a lattice L, then $P \cap Q$is a 2-absorbing ideal of L.

Proof. Let $x, y, z \in L$and $x \wedge y \wedge z \in P \cap Q$then $x \wedge y \wedge z \in P$and $x \wedge y \wedge z \in Q$. Since $P$and $Q$are prime ideals of L, we have, either (1)$x \wedge y \in P$or $z \in P$and $x \wedge y \in Q$or $z \in Q$, or (2)$x \wedge z \in P$or $y \in P$and $x \wedge z \in Q$or $y \in Q$, or (3)$y \wedge z \in P$or $x \in P$and $y \wedge z \in Q$or $x \in Q$. Without lost of generality, suppose that $x \wedge y \in P$or $z \in P$and $x \wedge y \in Q$or $z \in Q$. If $x \wedge y \in P$and $x \wedge y \in Q$then proof is obvious. If $z \in P$and $z \in Q$ then either $x \wedge z \in P$and $x \wedge z \in Q$or $y \wedge z \in P$and $y \wedge z \in Q$.

Similarly we have the following Lemma for weakly prime ideals.

Lemma 2.6: Let $P$and Qbe two distinct weakly prime ideals of a lattice $L$with zero, then $P \cap Q$is a weakly 2-absorbing ideal of L.

3. Results by Using Triple Zero

Definition 10: Let I be a weakly 2-absorbing ideal of a lattice L and $a,b,c \in L$. We say (a,b,c) is a triple zero of I if $a \wedge b \wedge c = 0$ then $a \wedge b \notin I$, $a \wedge c \notin I$ and $b \wedge c \notin I$.

Definition 11: For an ideal I of a lattice L, we define

(1) $I^2 = \{a \wedge b : a \neq b;\ a,b \in I\}$.

(2) $I^3 = \{a \wedge b \wedge c : a \neq b \neq c;\ a,b,c \in I\}$.

(3) $a \wedge b \wedge I = \{a \wedge b \wedge i : i \in I\}$.

(4) $a \wedge I^2 = \{a \wedge i \wedge j : i \neq j;\ i,j \in I\}$.

Definition 12: A lattice L is called modular if, for all elements $a,b,c \in L$, the following identity holds:
$(a \wedge c) \vee (b \wedge c) = [(a \wedge c) \vee b] \wedge c$.

Modular law: $a \leq c$ implies that
$a \vee (b \wedge c) = (a \vee b) \wedge c$.

Definition 13: A lattice L satisfying the following identities; for all $x,y,z \in L$,

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

is called distributive lattice.

Theorem 3.1: Let I be a weakly 2-absorbing ideal of a modular lattice L with zero and suppose that (a,b,c) is a triple zero of I for some $a,b,c \in L$. Then
$a \wedge b \wedge I = a \wedge c \wedge I = b \wedge c \wedge I = \{0\}$.

Proof. Suppose that $a \wedge b \wedge i \neq 0$ for some $i \in I$. Then
$0 \neq (a \wedge b \wedge c) \vee (a \wedge b \wedge i) \in I$
(since $a \wedge b \wedge i \neq 0$). As L is modular, we have
$a \wedge b \wedge (c \vee (a \wedge b \wedge i)) \in I$. Since (a,b,c) is a triple zero of I, $a \wedge b \notin I$. Therefore, either $a \wedge (c \vee (a \wedge b \wedge i)) \in I$ or
$b \wedge (c \vee (a \wedge b \wedge i)) \in I$.
As L is modular and $a \wedge b \wedge i \leq a$, we have
$a \wedge (c \vee (a \wedge b \wedge i)) = (a \wedge c) \vee (a \wedge b \wedge i) \in I$.
Similarly, $a \wedge b \wedge i \leq b$, we have
$b \wedge (c \vee (a \wedge b \wedge i)) = (b \wedge c) \vee (a \wedge b \wedge i) \in I$.
Thus, either $(a \wedge c) \vee (a \wedge b \wedge i) \in I$ or
$(b \wedge c) \vee (a \wedge b \wedge i) \in I$.
Now, we have $a \wedge c \leq (a \wedge c) \vee (a \wedge b \wedge i)$
and $b \wedge c \leq (b \wedge c) \vee (a \wedge b \wedge i)$.
Thus, either $a \wedge c \in I$ or $b \wedge c \in I$, which is a contradiction to (a,b,c) is a triple zero of I. Hence $a \wedge b \wedge I = \{0\}$. Similarly, we can show that $a \wedge c \wedge I = b \wedge c \wedge I = \{0\}$.

Theorem 3.2: Let I be a weakly 2-absorbing ideal of a distributive lattice L with zero and suppose that (a,b,c) is a triple zero of I for some $a,b,c \in L$. Then
$a \wedge I^2 = b \wedge I^2 = c \wedge I^2 = \{0\}$.

Proof. Suppose that $a \wedge x \wedge y \neq 0$ for some $x,y \in I$, $x \neq y$. As L is distributive, we have $a \wedge (b \vee x) \wedge (c \vee y) = [(a \wedge b \wedge c) \vee (a \wedge b \wedge y)] \vee [(a \wedge x \wedge c) \vee (a \wedge x \wedge y)]$. Since (a,b,c) is a triple zero of I, we have $a \wedge b \wedge c = 0$. By Theorem 3.1, we have
$a \wedge b \wedge i = a \wedge c \wedge i = b \wedge c \wedge i = \{0\}$, for $i \in I$. Thus $a \wedge (b \vee x) \wedge (c \vee y) = a \wedge x \wedge y \neq 0$.
As $0 \neq a \wedge x \wedge y \in I$, we have
$0 \neq a \wedge (b \vee x) \wedge (c \vee y) \in I$. Since I is a weakly 2-absorbing ideal of L, we have either $a \wedge (b \vee x) \in I$ or $a \wedge (c \vee y) \in I$ or $(b \vee x) \wedge (c \vee y) \in I$. Thus, either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$. Which is a contradiction to (a,b,c) is triple zero of I. Thus $a \wedge I^2 = \{0\}$. Similarly, we can show that $b \wedge I^2 = c \wedge I^2 = \{0\}$.

Theorem 3.3: Let L be a distributive lattice with zero. Let I be a weakly 2-absorbing ideal of L, that is not 2-absorbing ideal. Then $I^3 = \{0\}$.

Proof. Since I is not a 2-absorbing ideal of L, I has a triple zero (a,b,c) for some $a,b,c \in L$. We have

$$I^3 = \{x \wedge y \wedge z : x \neq y \neq z;\ x,y,z \in I\}.$$

Suppose that $x \wedge y \wedge z \neq 0$ for some $x,y,z \in I$. As L is distributive, we have

$$(a \vee x) \wedge (b \vee y) \wedge (c \vee z) =$$
$(a \wedge c \wedge b) \vee (x \wedge c \wedge b) \vee (a \wedge c \wedge y) \vee (x \wedge c \wedge y) \vee$
$(a \wedge z \wedge b) \vee (x \wedge z \wedge b) \vee (a \wedge z \wedge y) \vee (x \wedge z \wedge y)$.

By Theorem 3.1 and Theorem 3.2, we have $a \wedge b \wedge I = a \wedge c \wedge I = b \wedge c \wedge I = \{0\}$,
$a \wedge I^2 = b \wedge I^2 = c \wedge I^2 = \{0\}$ and since (a,b,c) is a triple zero of I, $a \wedge b \wedge c = 0$. Thus,
$(a \vee x) \wedge (b \vee y) \wedge (c \vee z) = x \wedge y \wedge z \neq 0$.
Hence $0 \neq (a \vee x) \wedge (b \vee y) \wedge (c \vee z) \in I$. As I is a weakly 2-absorbing ideal, we have, either $(a \vee x) \wedge (b \vee y) \in I$ or $(a \vee x) \wedge (c \vee z) \in I$ or $(b \vee y) \wedge (c \vee z) \in I$. Hence, either
$(a \wedge b) \vee (x \wedge b) \vee (a \wedge y) \vee (x \wedge y) \in I$ or
$(a \wedge c) \vee (x \wedge c) \vee (a \wedge z) \vee (x \wedge z) \in I$ or
$(b \wedge c) \vee (y \wedge c) \vee (b \wedge z) \vee (y \wedge z) \in I$.
Thus, either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$. Which is a contradiction to (a,b,c) is a triple zero of I. Hence $I^3 = \{0\}$.

Remark 3.1: The following example shows that the converse of above theorem does not hold.

Example 9: Consider the ideal $I = \{0,a,f\}$ of the lattice shown in Figure 4.

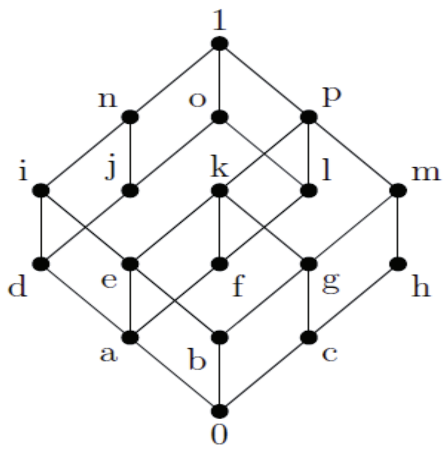

Figure 4

Here $I^3 = \{0\}$. Now for $n,o,p \in L$ and
$n \wedge o \wedge p = a \in I$ implies $n \wedge o = j \notin I$,
$n \wedge p = e \notin I$ and $o \wedge p = l \notin I$. Thus I is not a weakly 2-absorbing ideal of L.

Definition 14: Let A, B, C be ideals of a lattice L, we define following

$A^2BC = \{x \wedge y \wedge p \wedge q: x, y \in A;\ p \in B;\ q \in C;\ x \neq y\}$.

$AB^2C = \{p \wedge x \wedge y \wedge q: p \in A; x, y \in B;\ q \in C;\ x \neq y\}$.

$ABC^2 = \{x \wedge y \wedge p \wedge q: x \in A;\ y \in B;\ p, q \in C;\ p \neq q\}$.

$A^2B^2 = \{x \wedge y \wedge p \wedge q:\ x, y \in A;\ p, q \in B;\ x \neq y;\ p \neq q\}$.

$A^2C^2 = \{x \wedge y \wedge p \wedge q:\ x, y \in A;\ p, q \in C;\ x \neq y;\ p \neq q\}$.

$B^2C^2 = \{x \wedge y \wedge p \wedge q:\ x, y \in B;\ p, q \in C;\ x \neq y;\ p \neq q\}$

Theorem 3.4: Suppose that A, B, $C$are weakly 2-absorbing ideals of a distributive lattice $L$with zero such that none of them is a 2-absorbing ideal of L. Then
$A^2BC = AB^2C = ABC^2 = \{0\}$.

Proof. Suppose that $x \wedge y \wedge p \wedge q \neq 0$ for some $x, y \in A$ $(x \neq y)$ and $p \in B, q \in C$.
Hence $x \wedge y \neq 0$. As $A$is a weakly 2-absorbing ideal of $L$and is not 2-absorbing ideal of L, there exists a triple zero (a, b, c) of $A$for some $a, b, c \in L$. As $x \wedge y \in A$and (a, b, c) is a triple- zero of A, we have $(x \wedge y) \vee (a \wedge b \wedge c) \in A$. As $x \wedge y \neq 0$, $0 \neq (x \wedge y) \vee (a \wedge b \wedge c) \in A$. That is $0 \neq [(x \wedge y) \vee a] \wedge [(x \wedge y) \vee b] \wedge [(x \wedge y) \vee c] \in A$. Since $A$is a weakly 2-absorbing ideal, we have either $[(x \wedge y) \vee a] \wedge [(x \wedge y) \vee b] \in A$or $[(x \wedge y) \vee a] \wedge [(x \wedge y) \vee c] \in A$or
$[(x \wedge y) \vee b] \wedge [(x \wedge y) \vee c] \in A$. Hence either $a \wedge b \in A$or $a \wedge c \in A$or $b \wedge c \in A$, which is a contradiction to (a, b, c) is a triple zero of A. Hence $x \wedge y = 0$ and thus $x \wedge y \wedge p \wedge q = 0$. Thus$A^2BC = \{0\}$. Similarly, we can show that $AB^2C = ABC^2 = \{0\}$.

Theorem 3.5: Suppose that A, B, $C$are weakly 2-absorbing ideals of a distributivelattice $L$with zero such that none of them is a 2-absorbing ideal of L. Then
$A^2B^2 = A^2C^2 = B^2C^2 = \{0\}$.

Proof. We show that $A^2B^2 = \{0\}$. Suppose that $x \wedge y \wedge p \wedge q \neq 0$ for some $x, y \in A (x \neq y)$ and $p, q \in B$ $(p \neq q)$. Hence $x \wedge y \neq 0$. As $A$is a weakly 2-absorbingideal of $L$and $A$is not 2-absorbing ideal of L, there exists a triple zero (a, b, c) of$A$for some $a, b, c \in L$. As $x \wedge y \in A$and as (a, b, c) is a triple zero of A, we have
$(x \wedge y) \vee (a \wedge b \wedge c) \in A$. As $x \wedge y \neq 0$, we have $0 \neq (x \wedge y) \vee (a \wedge b \wedge c) \in A$. That is
$0 \neq [(x \wedge y) \vee a] \wedge [(x \wedge y) \vee b] \wedge [(x \wedge y) \vee c] \in A$.
Since $A$is a weakly 2-absorbing ideal, we have, either $[(x \wedge y) \vee a] \wedge [(x \wedge y) \vee b] \in A$or $[(x \wedge y) \vee a] \wedge [(x \wedge y) \vee c] \in A$ or
$[(x \wedge y) \vee b] \wedge [(x \wedge y) \vee c] \in A$. Hence, either $a \wedge b \in A$or $a \wedge c \in A$or $b \wedge c \in A$, which is a contradiction to (a, b, c) is a triple zero of A.
Hence $x \wedge y = 0$ and thus $x \wedge y \wedge p \wedge q = 0$. Thus $A^2B^2 = \{0\}$. Similarly we can show that $A^2C^2 = B^2C^2 = \{0\}$.

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