SOME PROPERTIES OF THE ATOM BASED GRAPH ON A LATTICE

S. K. NIMBHORKAR, U. R. BORSARKAR

Abstract: In this paper we associate a simple graph, denoted by $\Gamma_a(L^*)$, to a lattice L with a smallest element 0. The vertex set of $\Gamma_a(L^*)$, is $L^* = L - \{0\}$ and two distinct vertices are adjacent if and only if x ∧ y is an atom in L . We call this graph as the atom based graph of L. We study some properties of $\Gamma_a(L^*)$, and the graph $\Gamma_a(L^{**})$, whose vertex set is $L^{**} = L - \{0,1\}$.

Keywords: Atom, bipartite graph, complete graph, zero-divisor graph.

Introduction: The study of algebraic structures, using the properties of associated graphs is a topic of research in the last three decades. There are many papers on assigning a graph to a lattice see, for example [5,6,7,8].

In 1988, Beck [2] introduced the idea of a zero-divisor graph of a commutative ring R with identity. He defined $\Gamma_0(R)$, to be the graph whose vertices are all the elements of R and in which two vertices x and y are adjacent if and only if $xy = 0$. He was mostly concerned with coloring of $\Gamma_0(R)$. Let $\chi(R)$ and $\omega(R)$, respectively denote the chromatic number and the clique number of $\Gamma_0(R)$.

Beck conjectured $\chi(R) = \omega(R)$for any commutative ring R. This investigation of coloring of a commutative ring was then continued by Anderson and Naseer [1]. They gave a counter example to show that the above conjecture of Beck is not true.

Nimbhorkar, Wasdikar and Demayer [4] have shown that Beck's conjecture holds true for a meetsemilattice with zero.

The zero divisor graph of a lattice L was proposed and studied by Nimbhorkar et.al.[8]. In which the vertex set is the set of all elements in L and two vertices x, y are adjacent if and only if $x \wedge y = 0$.

A different method of associating a graph to a lattice Li.e incomparability graph of lattices was introduced and studied by Wasadikar and Survase [7,8] as follows. Let L be a lattice, let $W(L) = \{a \in L |$there exists$b \in L such that a \parallel b\}$. The incomparability graph of L, denoted by $\Gamma'(L)$, is a graph with vertex set $W(L)$ and two distinct elements $a, b \in W(L)$ are adjacent if and only if they are incomparable.

In this paper, we associate a simple graph $\Gamma_a(L^*)$to a lattice L with a smallest element 0, where the vertex set is $L^* = L - \{0\}$ and two distinct vertices in L^* are adjacent if and only if $x \wedge y$ is an atom in L. We call this as the atom basedgraph on L. We prove that this graph isalways connected and its diameter is either 1, 2 or 3 and its girth is either 3, 4 or ∞. We also show that it is never a complete graph. We show that a complete bipartite graph and a star graph can be realized as the atom based graph of a lattice.

Preliminaries: The concepts from lattice theory used in this paper are from Gr¨atzer [4] and those of graph theory can be found in West [13]. All the graphs are assumed to be simple and finite.

Throughout in this paper L denotes a finite lattice as a consequence L always contains an atom.

Definition 2.1: The zero element (unit element) of a lattice L (if it exists) is an element denoted by 0 (1) and is such that $0 \leq x$, $(x \leq 1)$for all$x \in L$.

Definition 2.2: Let $(P, \leq)$ be a poset with a smallest element 0. An element $x \in P$is called an atom if0 x, i.e. there is no $y \in L$ such that $0 < y < x$.

Definition 2.3: [8] Let L be a finite lattice with 0. Let $Z(L)$ be its set of all zero divisors and let $Z_0(L) = Z(L) - \{0\}$. A graph defined on L with the vertex set $Z_0(L)$and distinct$x, y \in Z_0(L)$are adjacent if and only if $x \wedge y = 0$ is called as the zero-divisor graph of L and denote it by $\Gamma(L)$.

Definition 2.4: The distance between distinct vertices x and$y$ of graph G is the length of the shortest path from x to y and it is denoted by $d(x, y)$.The diameter of a graph G is $Diam(G) = sup\{d(x, y) | x, y \in v(G)\}$.

Definition 2.5: The girth of a graph G is defined as the length of the shortest cycle in G. It is denoted by $gr(G)$.

Definition 2.6: Two graphs G and H are isomorphic, if there exists a bijection

$\varphi : V(G) \to V(H)$such that$uv \in E(G)$if and only if $\varphi(u)\varphi(v) \in E(H)$ for all $u, v \in G$.

3. Some results of $\Gamma_a(L^*)$

We denote the set of all atoms in L by $\Omega(L)$. We associate a simple graph with L, where the vertex set is $L^* = L - \{0\}$ and two distinct elements $x, y \in L^*$ are adjacent if and only if $x \wedge y \in \Omega(L)$. We denote this graph by $\Gamma_a(L^*)$, and call it the atom based graph on L.

The following example shows that this graph is different from the zero divisor graph, $\Gamma(L)$ and the incomparability graph, $\Gamma'(L)$ on L.

Example 3.1.

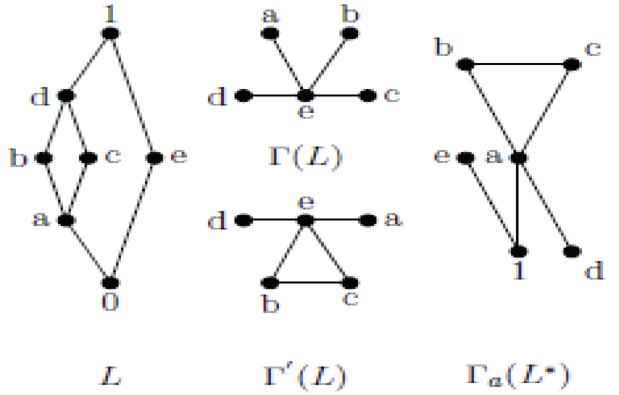

Figure 1

The following are some observations:

1. If $a \in \Omega(L)$, then for any $x \in a^u = \{x|x \in L and a \leq x\}, a \sim x$ and $x \nsim y$ for $y \notin a^u$.
2. Since L is a finite lattice $1 \in L$ and $1 \sim a, \forall a \in \Omega(L)$ and $1 \nsim x, \forall x \notin \Omega(L)$.
3. In $\Gamma_a(L^*)$,, $deg(1) = |\Omega(L)|$ and for every $a \in \Omega(L)$, $deg(a) = |a^u| - 1$.
4. If $a, b \in \Omega(L)$, then $a \nsim b$ in $\Gamma_a(L^*)$.
5. Two vertices a, b are adjacent in $\Gamma(L)$ if and only if a, b are nonadjacent in $\Gamma_a(L^*)$.

Example 3.2: For the lattice L shown in Figure 2, $\Gamma(L)$ and $\Gamma'(L)$ is the empty graph and $\Gamma_a(L^*)$,is the star graph.

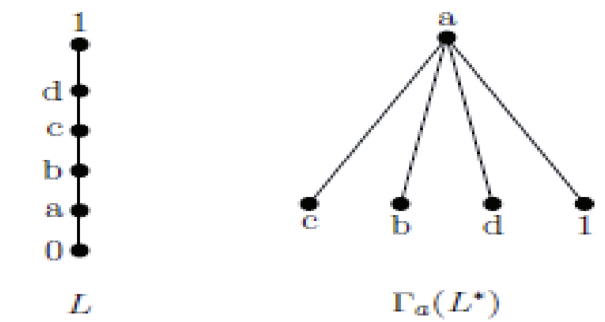

Figure 2

We note the following.

(1) $\Gamma(L)$ and $\Gamma'(L)$may be a null graph (with vertex set is empty), where as $\Gamma_a(L^*)$, is a nonnull graph.
(2) By the definition of $\Gamma(L)$ and $\Gamma'(L)$, and $\Gamma_a(L^*)$, we note that $1 \in L^*$but 1 does not belong to the vertex set of$\Gamma(L)$ and $\Gamma'(L)$. Hence the vertex sets of $\Gamma(L)$ and $\Gamma'(L)$are proper subsets of the vertex set of $\Gamma_a(L^*)$.
(3) We note that non-isomorphic lattices may have isomorphic $\Gamma_a(L^*)$, In the lattices shown in Example 3.2 and Example 3.3, $\Gamma_a(L^*) \cong \Gamma_a({L_1}^*)$ but$L \ncong L_1$

Example 3.3.

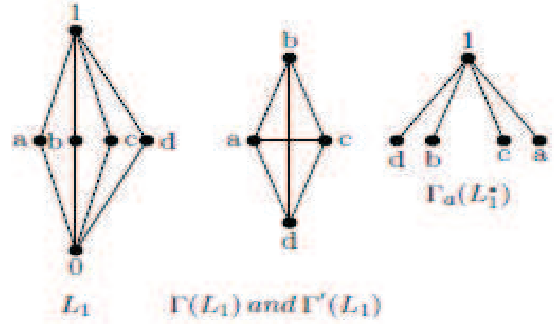

Figure 3

Theorem 3.1: $\Gamma_a(L^*)$, is always connected and $\text{diam}(\Gamma_a(L^*)) \leq 4$.

Proof: Let $x, y \in L^*$then either $(1) x, y \in \Omega(L)$ or $(2) x \in \Omega(L), y \notin \Omega(L)$or $(3) x, y \notin \Omega(L)$.
(1) Let $x, y \in \Omega(L)$. Then $x \sim 1 \sim y$ is a path in $\Gamma_a(L^*)$, hence $d(x, y) = 2$.
(2)Suppose that $x \in \Omega(L)$, $y \notin \Omega(L)$ then either $y \in x^u$or $y \notin x^u$. If $y \in x^u$ then $x \sim y$ in $\Gamma_a(L^*)$, and so $d(x, y) = 1$.If $y \notin x^u$ then as L is a finite lattice, there exists $c \in \Omega(L)$ such that $y \in c^u$ that is $y \sim c$, hence $y \sim c \sim 1 \sim x$ in $\Gamma_a(L^*)$, hence $d(x, y) = 3$.
(3) Suppose $x, y \notin \Omega(L)$ then either
(I) $x, y \in a^u$for some $a \in \Omega(L)$ or (II) x $\in a^u$ and y $\in b^u$ where $a, b \in \Omega(L)$.
(I) If $x, y \in a^u$ for some $a \in \Omega(L)$. Then either $x \wedge y = a$ or $x \wedge y \neq a$.
If $x \wedge y = a$, then $x \sim y$ and so $d(x, y) = 1$ and if $x \wedge y \neq a$, then $x \sim a \sim y$ in $\Gamma_a(L^*)$ so $d(x, y) = 2$.
(II) If $x \in a^u$ and $y \in b^u$ some $a, b \in \Omega(L)$ then $x \sim a, a \sim 1, 1 \sim b, b \sim y$ together imply $x \sim a \sim 1 \sim b\, y$. Therefore, $d(x, y) = 4$. Thus $\Gamma_a(L^*)$, is always connected and $\text{diam}(\Gamma_a(L^*)) \leq 4$.

Corollary 3.1: A disconnected graph cannot be realized as a $\Gamma_a(L^*)$, .

Now we give a characterization for $\Gamma_a(L^*)$,to be a star graph.

Definition 3.1: The lattice M_n is defined as a set $\{a_1, \ldots., a_n, 0, 1\}$, with $0 < a_i < 1$ and $a_i \parallel a_j$ for i ≠ j.

Definition 3.2: A lattice L is called as an integral lattice if | Ω(L) |= 1

Wasadikar and Survase [10] defined the linear sum of lattices as follows.

Definition 3.3: Let $L_1$and L_2 be two lattices, the linear sum of L_1 and L_2denoted by $L_1 \oplus L_2$ is obtained by placing the diagram of L_1 directly below the diagram of L_2 and adding a line segment from the maximum element of L_1 to the minimum element L_2.

Theorem 3.2:$\Gamma_a(L^*)$, is a star graph if and only if either (1) $L\ is\ M_n$ or (2) L is an integral lattice of the type $C_2 \oplus L_2$

Proof: Suppose that $\Gamma_a(L^*)$ is a star graph. We claim that the center of $\Gamma_a(L^*)$is either 1 or an atom of L.

If x is the center of $\Gamma_a(L^*)$, and $x \notin \Omega(L)$ or $x \neq 1$, then there exists $a \in \Omega(L)$ such that $x \in a^u$ and we have $x \sim a \sim 1$ is a path of length 2 in $\Gamma_a(L^*)$ and $x \nsim 1$, contradiction. Therefore, the center of $\Gamma_a(L^*)$,is either (A) $x = 1$ or (B) $x \in \Omega(L)$.

(A): If $x = 1$ is the center of $\Gamma_a(L^*)$, then for $y \in L^*$, $y \sim 1$ implies that $y \in \Omega(L)$, so $L - \{0,1\} = \Omega(L)$ which shows that L is M_n.

(B): Suppose that $x \in \Omega(L)$ and x is the center of $\Gamma_a(L^*)$. If $|\Omega(L)| \geq 2$, let $x, y \in \Omega(L)$ and $y \neq x$ then $y \nsim x$ but $x \sim 1 \sim y$ path of length 2, a contradiction to $\Gamma_a(L^*)$ is a star graph. Therefore, $|\Omega(L)| = 1$. Thus L is an integral lattice. Now we show that if $x \in \Omega(L)$ and $x \prec y$, then y is unique. If $x \prec z$, $z \neq y$, then $y \wedge z = x$, implies x, y, z forms a cycle of length 3 in $\Gamma_a(L^*)$ a contradiction to $\Gamma_a(L^*)$ is a star graph. Thus y is unique. Therefore L can be written as $C2 \oplus L2$ where $C2 = \{0, x\}$ and L_2 is a lattice with minimum element y.

Conversely, assume that L is M_n or L is an integral lattice of the type $C_2 \oplus L_2$. If L is M_n then $\Omega(L) = L - \{0,1\}$. Since $1 \sim x$ for all $x \in \Omega(L)$ and as $x \nsim y$ for all $x, y \in \Omega(L)$ implies $\Gamma_a(L^*)$,is a star graph with the center 1.

If L is an integral lattice of the type $C_2 \oplus L_2$, then the maximum element of C_2 is an atom of L say x and for every $y \in L_2$, $x \leq y$. Hence x is covered by the zero of L_2 implies for every pair $y, z \in L_2$, $x \sim y$ and $y \nsim z$ as $y \wedge z \neq x$. Which shows that $\Gamma_a(L^*)$, is a star graph with the center x.

Corollary 3.2: A star graph can be realized as a $\Gamma_a(L^*)$.

Theorem 3.3: A complete bipartite graph $K_{m,n}$is $\Gamma_a(L^*)$, for some finite lattice L if and only if in L, $x \notin \Omega(L)$ implies that $x \in \cap \{a_i^u : a_i \in \Omega(L)\}$.

Proof: Assume that $K_{m,n} = \Gamma_a(L^*)$, $m, n \geq 2$ for some finite lattice L. Let P_1 and P_2. be the two partitions of $K_{m,n}$. Since no two atoms are adjacent in $\Gamma_a(L^*)$ we may assume that P_1 = Ω(L). Hence the other partite set $P_2 \subseteq \{x \mid x \notin \Omega(L)\}$. Moreover for each $x \notin \Omega(L)$ is adjacent to some $a \in \Omega(L)$ implies $x \in P_2$ hence $\{x|x \notin \Omega(L)\}P_2$and so $\{x|x \notin \Omega(L)\} = P_2$. Since P_2 is partite set, no two $x, y \in P_2$ are adjacent to each other. Now we have for each $x \in P_2$ and for each $a \in P_1, x \sim a$,which implies $x \in a_i^u$, for every$a_i \in \Omega(L)$, therefore $x \in \cap a_i^u$. Conversely, assume that L is a finite lattice in which $\forall x \notin \Omega(L)$ implies $x \in a_i^u$ for every $a_i \in \Omega(L)$,. Since no two atoms are adjacent in $\Gamma_a(L^*)$, hence assuming $\Omega(L)$ as one partite set say P_1 and Considering other partite set $P_2 \subseteq \{x|x \notin \Omega(L)\}$. We claim that $x \nsim y$ if $x, y \in P_1$ or $x, y \in P_2$. If $x \sim y$for some $x, y \in L^*$ then clearly $x, y \notin P_1$. If $x, y \in P_2$ then $x \wedge y = a$ for some $a \in \Omega(L)$. By assumption $x, y \in a_i^u$ for every $a_i \in \Omega(L)$ implies $x \wedge y = a_i$, for each a_i is not possible. Now we have $x \sim a_i$for all $x \in P_2$ and all $a_i \in P_1$. Hence $\Gamma_a(L^*)$ is a complete bipartite graph $K_{m,n}$. where $m = |\Omega(L)|$ and $n = |P|$, where P = {x | x ∉ Ω(L)}.

Corollary 3.3: A complete bipartite graph can be realized as a $\Gamma_a(L^*)$.

Theorem 3.4: If $\Gamma_a(L^*)$, contains a cycle then $girth(\ \Gamma_a(L^*),\ ) \leq 4$.

Proof: Suppose that x ~ y ~ z ~ w ~· · · ~ x is the smallest cycle C of length n in$\Gamma_a(L^*)$and $n > 5$. We claim that C contains at least two atoms.

If C contains no atom or a single atom x, then $y \sim z$ in C implies that there exists atom in L such that $y \wedge z = a$, then $y \sim a\ z \sim y$ is a cycle of smallest length 3, a contradiction. Suppose C contains two atoms. As $a \nsim b$ for all $a, b \in \Omega(L)$ we may suppose that x, z ∈ Ω(L) or x, w ∈ Ω(L). If $x, z \in \Omega(L)$ then $x \sim 1, z \sim 1$ along with $x \sim y \sim z$ imply $x \sim y \sim z \sim 1 \sim x$ is a cycle of length 4, which is a contradiction. If $x, w \in \Omega(L)$ then $y \sim z$ implies that there exists $a \in \Omega(L)$ such that y ∧ z = a, so $y \sim a \sim z \sim y$ is a cycle of length 3, a contradiction. Therefore, $girth(\Gamma_a(L^*)) \leq 4$.

In the following two results we give a necessary and sufficient condition for girth($\Gamma_a(L^*)$) to be equal to 3 and 4.

Lemma 3.1: The girth of $\Gamma_a(L^*)$ is 3 if and only if there exists $a \in \Omega(L)$ and $b, c \in a^u$ such that $b \wedge c = a$.

Proof: If $a \in \Omega(L)$and $b \wedge c = a$ then a cycle $a \sim b \sim c \sim a$ is in $\Gamma_a(L^*)$. Conversely, if $x \sim b \sim c \sim x$ is in $\Gamma_a(L^*)$ then there exist $a \in \Omega(L)$ such that $b \wedge c = a$.

Theorem 3.5: The girth of ($\Gamma_a(L^*)$) is 4 if and only if $|\Omega(L)| > 2$ and there exists at least two atoms a, b in L such that $|a^u \cap b^u| > 2$, and there does not exist any $c \in \Omega(L)$ such that $x, y \in c_u$ and $x \wedge y = c$.

Proof: Suppose that girth($\Gamma_a(L^*)$) $= 4$. By Lemma 3.1, there does not exist any $c \in \Omega(L)$ such that $x, y \in c^u$ and $x \wedge y = c$. Now consider a cycle C, a ~ b ~ c ~ d ~ a of length 4 in $\Gamma_a(L^*)$. We claim that C contains exactly two atoms. If C contains no atom or only one atom say a, then $b \sim c$ implies $b \wedge c = e$, $e \in \Omega(L)$, then $b \sim c \sim e \sim b$ as a cycle of length 3, a contradiction to girth$(\Gamma_a(L^*)) = 4$. Also for $x, y \in \Omega(L), x \nsim y$ therefore, the cycle C contains exactly two atoms. Hence $|\Omega(L)| > 2$. Assume $a, c \in \Omega(L)$, since $b \sim a, b \sim c$, $d \sim a$, $d \sim c$ in C, implies $b, d \in a^u$ and $b, d \in c^u$. Hence we have $|a^u \cap c^u| > 2$. Conversely, assume that $|\Omega(L)| > 2$ and there exists at least two atoms $a, b \in L$ such that $|a^u \cap b^u| > 2$, and there does not exist any $c \in \Omega(L)$ such that $x, y \in c^u$and $x \wedge y = c$. Hence by Lemma 3.1, girth$(\Gamma_a(L^*)) \neq 3$.Now consider two atoms $a, c \in L$ such that $|a^u \cap c^u| > 2$, let $b, d \in a^u \cap c^u$,with $b \wedge d \notin \Omega(L)$. Then $b \sim a$, $b \sim c, d \sim a$, $d \sim c$, $b \nsim d$in$\Gamma_a(L^*)$. Therefore, $a \sim b \sim c \sim d \sim a$is a cycle of length 4, so girth$(\Gamma_a(L^*)) = 4$.

Lemma 3.2:$\Gamma_a(L^*)$is never a complete graph, except the chain C_3.

Proof: If$b \notin \Omega(L)$, then we have $1 \nsim b$,in $\Gamma_a(L^*)$. Thus $\Gamma_a(L^*)$can never be a complete graph. For the chain C_3, we have $\Gamma_a({C_3}^*)$. is the path P_2, a complete graph.

4. Some results of$\Gamma_a(L^{**})$

In previous section we considered the vertex set as $L^* = L - \{0\}$, and showed that $\Gamma_a(L^*)$is always connected as there exist a path between any two atoms of Lpassing through 1. In this section weconsider the vertex set as $L^{**} = L - \{0,1\}$ and obtain some result on $\Gamma_a(L^{**})$.

Lemma4.1: If L is a finite lattice consisting of chains C_i, between 0 and 1. $1 \leq i \leq n$, such that $C_i \cap C_j = \{0,1\}$, $(C_i \neq \{0,1\})$, for every $i, j \in \{1,2,3,\ldots\ldots,n\}$, $i \neq j$, then $\Gamma_a(L^*)$ is a tree and $\Gamma_a(L^{**})$is a disconnected graph with n number of star components.

Proof: Let L be a finite lattice consisting of chains C_i, between 0 and 1. $1 \leq i \leq n$, such that $C_i \cap C_j = \{0,1\}$, $(C_i \neq \{0,1\})$, for every $i, j \in \{1,2,3,\ldots\ldots,n\}$, $i \neq j$, Clearly for each $a \in \Omega(L)$ there is only one chain C_i of L such that $a \in C_i$, and as for every i, C_i, is an integral lattice of the type $C_2 \oplus L_2$, where $C_2 = \{0, a\}$ by Theorem 3.2 $\Gamma_a({C_i}^*)$. and hence $\Gamma_a({C_i}^{**})$ are star graphs with a as the center. Since $C_i \cap C_j = \{0,1\}$, for every $i \neq j$, we have $1 \in \cap C_i$ hence $1 \in \cap \Gamma_a({C_i}^*)$, $1 \leq i \leq n$, which shows that $\Gamma_a({C_i}^*)$is a tree with 1 as the root node. As $1 \notin L^{**}$ hence $\Gamma_a({C_i}^{**})$, $1 \leq i \leq n$ is a disconnected graph containing n number of star components.

Example 4.1: $L$is a finite lattice consisting of 4 chains $C_i$between 0 and 1. $1 \leq i \leq 4$, such that $C_i \cap C_j = \{0,1\}$, $(C_i \neq \{0,1\})$, for every $i, j \in \{1,2,3,4\}$, $i \neq j$, then $\Gamma_a(L^*)$ is a tree and $\Gamma_a(L^{**})$is a disconnected graph with 4 number of star components.

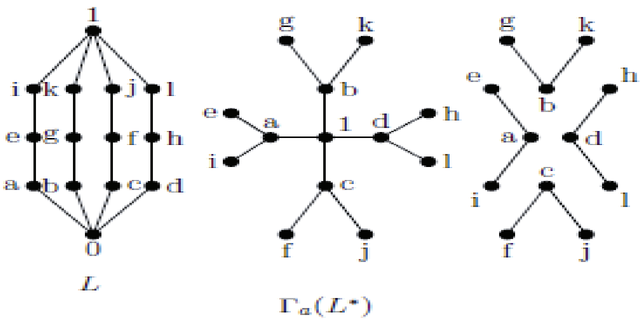

Figure 4

In previous section we prove that $\Gamma_a(L^*)$is never a complete graph. In the next theorem we give a characterization for $\Gamma_a(L^{**})$to be a complete graph.

Theorem 4.1: $\Gamma_a(L^{**})$is a complete graph, if and only if $L$is a finite integral lattice and for atom $a \in L$and for all $x \in a^u - \{1\}, x \prec 1$.

Proof: Let $\Gamma_a(L^{**})$is a complete graph. If $a, b \in \Omega(L)$ then $a \nsim b$therefore, $|(L)| = 1$ hence L is an integral lattice.

Moreover if $a \in \Omega(L)$then for all $x \in a^u - \{1\}$, $x \prec 1$otherwise if for some $y \in a^u - \{1\}$, $y \nprec 1$ then there exists $z \in a^u - \{1\}$, and $y \prec z$. This implies $y \nsim z$in $\Gamma_a(L^{**})$ which contradicts the completeness of $\Gamma_a(L^{**})$.

Conversely assume that $L$is an integral lattice and for all $x \in a^u - \{1\}, x \prec 1$, We have for every $x, y \in a^u - \{1\}$, $x \sim y$ as $x \wedge y = a$ and $x \sim a$ which shows that $x \sim y$, for every $x, y \in L^{**}$ hence $\Gamma_a(L^{**})$ is a complete graph .

Lemma 4.2: If L is an integral lattice then $\Gamma_a(L^{**})$ is always connected and there exists at least one pendent vertex in $\Gamma_a(L^*)$.

Proof: If $|(L)| = 1$, then for $a \in \Omega(L)$ and for all $x, y \in L^{**} - \Omega(L)$, we have $x \sim a \sim y$ is in $\Gamma_a(L^{**})$implies $\Gamma_a(L^{**})$ is connected.

Since $deg(1) = 1$ in $\Gamma_a(L^*)$ implies 1 is a pendent vertex in $\Gamma_a(L^*)$.

Corollary 4.1: If a disconnected graph is realizable as a $\Gamma_a(L^{**})$ then L cannot be an integral lattice.

Example 4.2: An example of a non integral lattice whose $\Gamma_a(L^{**})$ is disconnected.

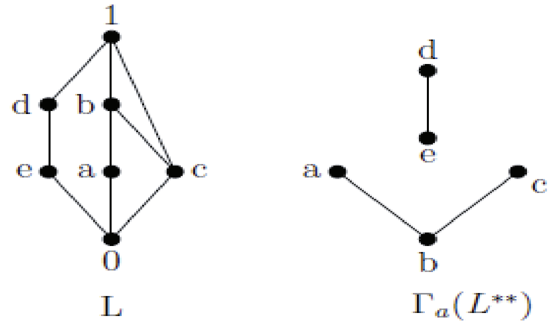

Figure 5

Example4.3: An example of a non integral lattice whose $\Gamma_a(L^{**})$is connected.

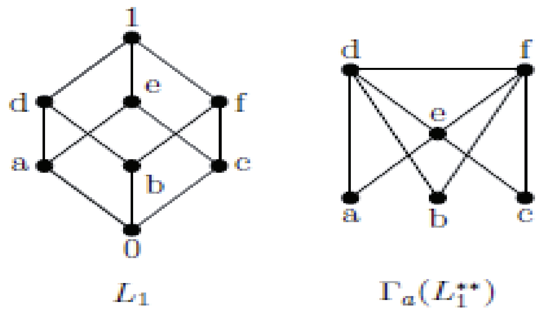

Figure 6

From Figure 5 and Figure 6 we observed that even if$L$is not an integral lattice then $\Gamma_a(L^{**})$ may be a connected graph or disconnected graph. Hence in the next theorem we give a characterization of connectedness of $\Gamma_a(L^{**})$.

Theorem 4.2: $\Gamma_a(L^{**})$ is connected if and only if (1) For all $a, b \in \Omega(L)$, $a^u \cap b^u \neq \emptyset$ or (2) If $a^u \cap b^u = \emptyset$ for some $a, b \in \Omega(L)$, then there exists $c \in \Omega(L)$,such that $a^u \cap c^u \neq \emptyset$ and $c^u \cap b^u \neq \emptyset$.If $a^u \cap c^u = \emptyset$(or if $b^u \cap c^u = \emptyset$) then there exists $d \in \Omega(L)$ such that $a^u \cap d^u \neq \emptyset$, $d^u \cap c^u \neq \emptyset$ and $b^u \cap c^u \neq \emptyset$.

Proof: Suppose that $\Gamma_a(L^{**})$ is connected. Clearly the theorem holds trivially for an integral lattice, so we assume that $|(L)| \geq 2$.
Also for all $a, b \in \Omega(L)$, $y \nsim z$in $\Gamma_a(L^{**})$ implies $d(a, b) \geq 2$.
We use mathematical induction on k, where $k = min\{d(a, b) \mid a, b \in \Omega(L)\}$.
If $k = 2$, then for$a, b \in \Omega(L)$ there exists a path$a \sim x \sim b$ in $\Gamma_a(L^{**})$hence $x \in a^u \cap b^u$ which implies $a^u \cap b^u \neq \emptyset$and this holds for all $a, b \in \Omega(L)$. Hence the statement holds for $k = 2$.
If $k = 3$, then for $a, b \in \Omega(L)$ there exists a path $a\sim x\sim y\sim b$in $\Gamma_a(L^{**})$. Clearly $x \in a^u$but $x \notin b^u$, and $y \in b^u$but $y \notin a^u$, also there exists $c \in \Omega(L)$ such that $x \wedge y = c$ implies $x \in a^u \cap c^u$ and $y \in c^u \cap b^u$. There fore $a^u \cap c^u \neq \emptyset$and $c^u \cap b^u \neq \emptyset$. Hence the statement holds for $k = 3$.
Now we assume that the statement holds for $k = n$.
If $k = n + 1$, then for $a, b \in \Omega(L)$ there exist a path$a \sim x_1 \sim x_2 \sim \cdot\cdot \sim x_n \sim b$ in $\Gamma_a(L^{**})$ Therefore, we have $c_1, c_2, \cdots c_{n-1}$ in $\Omega(L)$, such that $x_1 \in a^u \cap c_1^u$, $x_2 \in c_1^u \cap c_2^u$, $x_3 \in c_2^u \cap c_3^u$, $..x_{n-1} \in c_{n-2}^u \cap c_{n-1}^u$. Either $x_n \in b^u \cap c_{n-1}^u$ or $x_n \notin b^u \cap c_{n-1}^u$.
If $x_n \in b^u \cap c_{n-1}^u$then obviously the statement holds for $k = n + 1$. If $x_n \notin b^u \cap c_{n-1}^u$ then $x_{n-1} \sim x_n$implies that there exists $c_n \in \Omega(L)$ such that $x_{n-1} \wedge x_n = c_n$, so $x_{n-1} \in c_{n-1}^u \cap c_n^u$ and $x_n \in c_n^u \cap b^u$implies $a^u \cap c_1^u \neq \emptyset$, $c_1^u \cap c_2^u \neq \emptyset$, $c_2^u \cap c_3^u \neq \emptyset \cdots c_{n-1}^u \cap c_n^u \neq \emptyset, c_n^u \cap b^u \neq \emptyset$so the statement holds for $k = n+ 1$. Hence by the mathematical induction we conclude that the statement holds for all n. Conversely suppose that the assumption holds or L. If $L$is a finite integral lattice then by Lemma 4.2, $\Gamma_a(L^{**})$ is connected. Therefore, we suppose that$|(L)| \geq 2$. Let$x, y \in L^{**}$and $x, y \notin \Omega(L)$, then there exists $a, b \in \Omega(L)$ such that$x, y \in a^u(or x, y \in b^u) or\ x \in a^u and y \in b^u$.
If $x, y \in a^u$then $x \sim a \sim y$is a path in $\Gamma_a(L^{**})$ and if $x \in a^u and\ y \in b^u$ implies $x \sim a$, $b \sim y$ is in $\Gamma_a(L^{**})$. So for the connectedness of $\Gamma_a(L^{**})$ it is sufficient to prove that there exists a path between any two atoms.
(a) Let $a, b \in \Omega(L)$, and if $x \in a^u \cap b^u$then $a \sim x \sim b$is a path in $\Gamma_a(L^{**})$.
(b) Now suppose that $a^u \cap b^u = \emptyset$ for some $a, b \in \Omega(L)$, then there exists $c \in \Omega(L)$, such that $a^u \cap c^u \neq \emptyset$ and $c^u \cap b^u \neq \emptyset$. Therefore for $x \in a^u \cap c^u$ and $y \in c^u \cap b^u, a \sim x \sim y \sim b$ is a path in $\Gamma_a(L^{**})$. If $a^u \cap c^u = \emptyset$ (or if $b^u \cap c^u = \emptyset$) then there exists $d \in \Omega(L)$such that $a^u \cap d^u \neq \emptyset$, $d^u \cap c^u \neq \emptyset$ and $b^u \cap c^u \neq \emptyset$. Then for $x \in a^u \cap d^u$ and $y \in d^u \cap c^u$, $z \in c^u \cap b^u$ a path $a \sim x \sim d \sim y \sim c \sim z \sim b$ is in $\Gamma_a(L^{**})$.

Theorem 4.3: If $L \cong L_1$then $\Gamma_a(L^*) \cong \Gamma_a(L_1{}^*)$so also $\Gamma_a(L^{**}) \cong \Gamma_a(L_1{}^{**})$.

Proof. Let L, L_1be the lattices such that $L \cong L_1$. Let $\Gamma_a(L^*)$and $\Gamma_a(L_1{}^*)$ be the atom based graph of lattices L, L_1. Assume $\phi: L \to L_1$the lattice isomorphism. For $x \in L$ define $\psi: \Gamma_a(L^*) \to \Gamma_a(L_1^*)$ by $\psi(x) = y$ where $y = \phi(x)$. As $L \cong L_1$ so $|L| = |L_1|$hence $|V(\Gamma_a(L^*))| = |V(\Gamma_a(L_1{}^*))|$. Now consider $a_1 \sim a_2$ in $\Gamma_a(L^*)$ then $a_1 \wedge a_2 = c_1$ where $c_1 \in \Omega(L)$. By the definition of $\psi(x) = y$, there exists b_1, b_2, $d_1 \in L_1$ such that $\phi(a_1) = b_1$, $\phi(a_2) = b_2$, $\phi(c_1) = d_1$. We claim that $d_1 \in \Omega(L_1)$. Otherwise for $d_1 \notin \Omega(L_1)$ there exist $f_1 \in \Omega(L_1)$ such that $\phi(e_1) = f_1$ and $0' \leq f_1 \leq d_1$ that is $\phi(0) \leq \phi(e_1) \leq \phi(c_1)$, which implies $0 \leq e_1 \leq c_1$, so $c_1 \notin \Omega(L)$ a contradiction.

Now $b_1 \wedge b_2 = \phi(a_1) \wedge \phi(a_2) = (a_1 \wedge a_2) = \phi(c_1) = d_1$ where $d_1 \in \Omega(L_1)$ implies $b_1 \sim b_2$ in $\Gamma_a(L_1{}^*)$. Therefore, for $a_1 \sim a_2$ in $\Gamma_a(L^*)$, $\psi(a_1) \sim \psi(a_2)$ that is $b_1 \sim b_2$ in $\Gamma_a(L_1{}^*)$. Hence $\Gamma_a(L^*) \cong \Gamma_a(L_1{}^*)$ so also $\Gamma_a(L^{**}) \cong \Gamma_a(L_1{}^{**})$.

In the lattice shown in Example 3.2 and Example 3.3, $\Gamma_a(L^*) \cong \Gamma_a(L_1{}^*)$ but $L \ncong L_1$. Therefore converse of the Theorem 4.3 not hold.

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