ON COMMUTATIVITY OF SEMIPRIME RINGS WITH (*A*, *B*) DERIVATIONS

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Abstract: If D is an (α,β) derivation of a 2- torsion free semiprimering, U a nonzero ideal of R and $[D(x),D(y)] + xy = 0$ for all x,y in U, then R contains a nonzero central ideal. Also if R admits a nonzero (α,β) Derivation D such that $D(x)D(y)+ D(xy)=D(y)D(x)+D(yx)$ for all x,y in U, where U is a nonzero left ideal, then $D(U)$ centralizes $[U,U]$. These results are extensions of the results shown by Herstein and Daif in terms of (α,β) derivations.

Key words: Center, central ideal, commutator, derivation, semiprime ring, (α, β) derivation.

Introduction: In his note on derivations, Herstein[9] showed that if a 2-torsion free prime ring R admits a nonzero derivation D such that $[D(x),D(y)]=0$ for all x, y in R, then R is commutative and Daif [9 and 7] extended this theorem in the situation when the ring is semiprime and the condition $[D(x),D(y)]=0$ is merely satisfied on the ideal of the ring. Aydin[1] generalized the theorems of Herstein and Daif for an ideal U and (α,β) derivation of R. He also generalized the results of Bresar and Vukman[6] by taking an ideal of R instead of R and extended to more general mappings.

In this paper we extend the results of Herstein and Daifin terms of (α,β) derivations and prove that if D is a nonzero (α,β) derivation on a semiprime ring R with a nonzero ideal. A ring R is said to be prime if $xay=0$ implies $x=0,y= 0$, for all $x,a,y\in R$ and R is semiprime if $xax=0$ implies $x=0$, for all $x,a\in R$. An additive mapping D from a ring R to R is a (α,β) derivation if $D(xy)=D(x)\alpha(y)+\beta(x)D(y)$ for all $x,y\in R$. The center Z of R is defined as $Z=\{z\in R/[z,R]=0\}$.

A commutator $[x,y]$ is defined as $xy-yx$ and an anticommutator $(x o y)$ is defined as $xy+yx$.

We will make an extensive use of the commutator identities $[x,yz]=y[x,z]+[x,y]z$ and $[xy,z]=x[y,z]+[x,z]y$.

Preliminaries: We start with few technical Lemmas. The proof of these may be found in [8].

Lemma1 : *Let R be a semiprime ring and U a non zero left ideal. If D is an (α, β) derivation of R, which is centralizing on U, then D is commuting on U.*♦

Lemma 2 : *Let R be a semiprime ring and U a nonzero left ideal . If R admits an (α,β) Derivation D which is non-zero and centralizing on U, then R contains a nonzero centralideal.*♦

Lemma 3: *Let R be a semiprime ring and U a nonzero two sided ideal of R. If $x\in R$ and x centralizes $[U, U]$, then x centralizes U.* ♦

Lemma 4: *Let R be a semiprime ring and U a nonzero ideal of R. Let $[U, U] = \{[x, y] / x, y\in U\}$. If $Z\in R$ and Z centralizes $[U, U]$ then Z centralizes U.*♦

Lemma 5: *If R is a prime ring, the centralizer of any one-sided ideal is equal to the center of R.*♦

Main Section:

Theorem 1: *Let R be a 2 - torsion free semiprime ring and U a nonzero two sided ideal of R. If R admits (α, β) derivation D which is nonzero on U and $[D(x), D(y)] + xy = 0$ for all $x, y\in U$, then R contains a nonzero central ideal.*

Proof : Given $0 = [D(x),D(y)] + xy,$ (1)

for all $x, y\in U$. Replacing y by yz, we obtain

$0 =D(y)[D(x), \alpha(z)] + [D(x), D(y)] \alpha(z) + \beta(y)[D(x), D(z)] + [D(x), \beta(y)]D(z) +xyz.$

Now replacing $\alpha(z)$ by z we obtain

$0=D(y)[D(x), z] + \beta(y)[D(x), D(z)] + \{[D(x), D(y)] + xy\}z + [D(x), \beta(y)]D(z)$

By using (1)

$0=D(y)[D(x),z] +\beta(y)(-xz) + [D(x), \beta(y)]D(z).$

Therefore

$0 =D(y)[D(x), z] -xz\beta(y) + [D(x), \beta(y)]D(z)$ (2)

By putting $z= zr$ where $r \in R$ we obtain

$0=D(y)z[D(x),r] +D(y)r[D(x),z] -\beta(y)xzr+[D(x), \beta(y)]\beta(z)D(r) + [D(x), \beta(y)]D(z)\alpha(r),$

Replacing $\alpha(r)$ by r we obtain

$0 =D(y)z[D(x), r] +r(D(y)[D(x), z] -xz\beta(y)) + [D(x), \beta(y)]D(z)r+ [D(x), \beta(y)]\beta(z)D(r)$

Using (2) it reduces to

$0 =D(y)z[D(x), r] + [D(x), \beta(y)]\beta(z)D(r).$ (3)

Now substituting $r = D(x)$, $x\in U$ in (3) to get

$0 =[D(x),\beta(y)]\beta(z)D^2(x).$ (4)

for all $x, y, z \in U$.

Now from (3.4), for all $[D(x), \beta(y)]\beta(z)RD^2(x)= \{0\}$, for all $x, y, z\in U$.

Hence for each P_α, we either we have

(a) $[D(x), \beta(y)]U\subseteq P_\alpha$, for all $x, y\in U$ or

(b) $D^2(U) \subseteq P_\alpha$.

We note that $[D(x), \beta(y)]RU\subseteq P_\alpha$ for each (a) prime P_α, so either $[D(x), \beta(y)] \in P_\alpha$ for all $x, y\in U$ or $U\subseteq P_\alpha$.

In either case $[D(x), \beta(y)]\in P_\alpha$, (5)

for all $x, y\in U$ and all prime P_α.

Now we consider P_α as prime ideals. Taking $x, y\in U$, we have $D^2(xy) =D^2(x) \alpha^2(y) + \beta^2(x)D^2(y) + 2\alpha\beta D(x)D(y) \in P_\alpha$.So $2\alpha\beta D(x)D(y)\in P_\alpha$, for all $x,y\in U$. Replacing y by zy for all $x, y, z \in U$, $2\alpha\beta D(x)D(zy) \in P_\alpha$, $2\alpha\beta D(x)[D(y)\alpha(y) + \beta(z)D(y)]\in P_\alpha$.

So $2\alpha\beta D(x)D(z)D(y)\in P_\alpha$. (6)

Hence $2\alpha\beta D(x)D(z)RD(y) \in P_\alpha$ and

$2\alpha\beta D(x)D(z)RD(y) \in P_\alpha$. (7)

It follows that either $D(U)\subseteq P_\alpha$ or $2\alpha\beta D(x)\alpha(y)$ and $2\beta(y)D(x) \in P_\alpha$.
In either case, $2[D(x), \beta(y)] \in P_\alpha$. (8)
for all $x, y\in U$ and (b) prime P_α. Thus for all $x, y\in U$ we have that $2[D(x), \beta(y)] \in\cap_\alpha P_\alpha\{0\}$ by (5) and (8). Since R is 2- torsion free,$[D(x), \beta(y)] = 0$.In particular $[D(x), x] = 0$ for all $x\in U$. So the Theorem follows by the Lemma 3.♦

Theorem 2: *Let R be a semiprime ring and U be a nonzero left ideal of R. If R admits a nonzero* (α, β) *derivation D such that* $D(x)D(y) + D(xy) = D(y)D(x) + D(yx)$ *for all* $x, y\in U$, *then* $D(U)$ *centralizes* $[U, U]$.

Proof: We have $D(x)D(y) + D(xy) = D(y)D(x) + D(yx)$ for all $x, y\in U$.
This implies,
$[D(x), D(y)] = [D(y), x] + [y,D(x)]$. (9)
for all $x, y\in U$. Substituting yx for y, we get $[D(x), D(yx)] = [D(yx),x] + [yx,D(x)]$.
Hence $D(y)[D(x),\alpha(x)] + [D(x), \beta(y)]D(x) = [\beta(y), x]D(x)$. (10)
for all $x, y\in U$.
Now replacing by xy we get
$D(xy)[D(x), \alpha(x)] + [D(x), \beta(xy)]D(x) = [\beta(yx), x]D(x)$.
i.e., $D(x)\alpha(y)[D(x), \alpha(x)] + \beta(y)[D(x), \beta(x)] + \beta(x)[D(y)[D(x), \alpha(x)] + D(x)[D(x), \beta(y)] = [\beta(y), x] \beta(x)D(x)$.
Using equation (10) it reduces to
$D(x)\alpha(y)[D(x), \alpha(x)] + [D(x), \beta(x)] \beta(y)D(x) = 0$, (11)
For all $x, y\in U$.
By replacing y by $D(x)y$ in (10)
$D(D(x)y)[D(x), \alpha(x)] + [D(x), \beta(D(x)y]D(x) = [\beta(D(x)y, x]D(x)$,
i.e.,$D^2(x)\alpha(y)[D(x),\alpha(x)]+\beta(D(x))[D(y)[D(x), \alpha(x)] + D(x)[D(x), \beta(y)]]=\beta D(x)D(x)[\beta(y), x] + [\beta D(x), x] \beta(y)D(x)$.Using equation(10) we get $D^2(x)\alpha(y)[D(x), \alpha(x)] = [\beta D(x), x] \beta(y)D(x)$.If we assume that $\beta D(x) = D(x)\beta$ then in the above relation we get $D^2(x)\alpha(y)[D(x), \alpha(x)] = [D(x), \beta(x)]\beta(y)D(x)$.
Hence
$D^2(x)\alpha(y)[D(x), \alpha(x)] - [D(x), \beta(x)]\beta(y)D(x) = 0$. (12)
From the relations equations (11) and (12) we obtain
$D^2(x)\alpha(y)[D(x),\alpha(x)] + D(x) \alpha(y)[D(x), \alpha(x)] = 0$.
$[D^2(x)+ D(x)]\alpha(y)[D(x), \alpha(x)] = 0$, (13)
for all $x,y\in U$. Thus the relation (13) yields
$[D^2(x) + D(x)]RU[D(x), \alpha(x)] =\{0\}$, (14)
for all $x, y \in U$. Since R is semiprime, it has a family $P= \{P_\alpha :\alpha\in\wedge\}$ of prime ideals such that $\cap_\alpha P_\alpha = \{0\}$.
Let P be a typical one of these by the relation equation (14) we have that for each $x\in U$, either $U [D(x), \alpha(x)] \subseteq P$ or $D(x) + D^2(x) \in P$ for all $x, y\in U$. Now we use the kind of argument employed in the Theorem(1) in effect performing calculation of modulo P in [2]. Then we get the conclusion that either
$UD(U) \subseteq P$ or$[x+ D(x), R] \subseteq P$. (15)
In the first case, again employing the argument of [2] modulo P, we obtain that either
$U\subseteq P$ or $[D(x),D(t)]\in P$, (16)
for all $x,t \in U$.
Now we assume that $[x + D(x),R] \subseteq P$, then we $[x,D(t)]+[D(x),D(t)] \subseteq P$, for all $x,t\in U$.
But from the relation(9) we have$[D(x),D(t)]+[x ,D(t)] =[t,D(x)]$.
Hence we have $[t,D(x)]\in P$, (17)
for all $x,t\in U$. Putting $t=D(y)$ we get, $[D(y)t,D(x)]\in P$, for all $x,y,t\in U$. By using the relation (17) we get $D(y)[t,D(x)] + [D(y),D(x)]t \in P$, for all $x,y,t \in U$.
$[D(y),D(x)]t\in P$, (18)
For all $x,y,t \in U$. By the relation (18), we have $[D(y),D(x)]RU\subseteq P$, for all $x, y \in U$. Consequently either $U\subseteq P$ or $[D(x),D(t)] \in P$, for all $x, t\in U$. Thus in any event, for each $x, t\in U$ we have either $U \subseteq P$or $[D(x),D(t)] \in P$ which are the same alternatives as in the relation (16). If we consider the case $U\subseteq P$, then from the relation (9) we get $[D(x),D(t)] \in P$, for all $x, t\in U$. Since $\cap_\propto P_\propto=\{0\}$, we conclude that $[D(x),D(t)]=0$ for all $x,t\in U$. From the relation (9), we get $[D(t),x]+[t, D(x)]=0$. That is, $D(t)\alpha(x)+\beta(t)D(x)=D(x)\alpha(t)+\beta(x)D(t)$, which is nothing but $D(tx)=D(xt)$, for all $x,t\in U$. This implies that $D[x,t]=0$ for all $x,t\in U$. But $D([x,t]z)=D(z[x,t])$. Hence $[x,t]D(z)= D(z)[x,t]$ for all $x, z,t\in U$. Thus $D(U)$ centralizes $[U,U]$ as required.♦

Theorem 3: *Let R be a prime ring and U a nonzero two sided ideal. If R admits a nonzero* (α,β) *derivation such that* $D(xy)=D(yx)$ *for all* $x,y\in U$, *then R is commutative.*

Proof: Let $c\in U$ be a constant that is an element such that $D(c) =0$. Let z be an arbitrary element of U. Then by the condition we get $D(cz)=D(zc)$ that is $D(c)\alpha(z)+\beta(c)D(z)=D(z)\alpha(c)+\beta(z)D(c)$.Since$D(c)=0$, this implies, $\beta(c)D(z)=D(z)\alpha(c)$. Now for each $x,y\in U,[x,y]$ is constant.
Hence $D[x,y]=[x,y]D(z)$, (19)
For all $x,y,z\in U$. By using Lemmas (4) and (5) we obtain $D(z)$ is central for all $z\in U$.♦

Theorem 4: *Let R be a2-torsion prime ring and let U be a nonzero left ideal. If D is a nonzero* (α,β) *derivation such that* $D(xy)=D(yx)$ *for all* $x,y\in U$ *then either R is a commutative or* $D^2(U)=\{0\}=D(U)D(U)$.

Proof: We have $D(xy)=D(yx)$. That is $[x,D(y)]=[y,D(x)]$. Replacing y by y^2, we get $[x,D(y^2)]= [y^2,D(x)]$, i.e.,
$D(y)[x,\alpha(y)]+[x,D(y)]\alpha(y)]+\beta(y)[x,D(y)]+ [x,\beta(y)]D(y)= y[y,D(x)]+[y, D(x)]y$.
Now we replacing $\alpha(y) = y,\beta(y) = y$ in above equation we obtain $D(y)[x,y]+[x,D(y)]y+ y[x,D(y)]+ [x,y]D(y)= y[y,D(x)]+[y,D(x)]y$.
i.e., $D(y)[x,y]+[x,y]D(y)+([x,D(y)] - [y,D(x)])y +y([x,D(y)] -[y,D(x)])=0$.
By the hypothesis introduces to

$D(y)[x,y] + [x, y]D(y) = 0.$ (20)
Recalling the relation (19) and using the fact that R is 2 –torsion free, we obtain
$D(y)[x,y]=0$ and $[x,y]D(y)=0$, (21)
For all $x,y\in U$.
In all second of equalities replacing x by xw, where $w\in U$, we obtain $[x,y]wD(y)+w[x,y]D(y)=0$. This implies $[x,y]wD(y)=0$. Hence $[x,y]UD(y)=\{0\}=[x, y]RUD(y)$, for all $x,y\in U$. Since $D\neq 0$ we can conclude that $[x,y]=0$ or $UD(y)=\{0\}$.That is either x is central or $UD(y)=\{0\}$.On the other hand, the first equality of the relation (21) yields $D(y)\ U[x,y]=0= D(y)RU[x,y]$, for all $x,y\in U$.
Since $D\neq 0$ we get $U[x,y]=\{0\}$, (22)
for all $x,y\in U$. We assume that R is not commutative and hence U is not central. By the relations (21) and (22) we have $UD(U) = \{0\}$ and $U\ [x,y]=\{0\}$, for all $x,y\in U$. Now $D(xy)=D(yx)$ implies that $D(x)\alpha(y)+\beta(x)D(y)=D(y)\alpha(x)+ \beta(y)D(x)$. Using the above conditions, it reduces to
$D(x)\alpha(y)=\beta(y)D(x)$, (23)
for all $x,y\in U$. Replacing y by ry for arbitrary $r\in R$ we get $D(x)\alpha(ry)=D(ry)\beta(x)$. Thus $D(x)\alpha(ry)= [D(x)\alpha(y)+ \beta(r)D(y)]\ \beta(x)$. Hence $D(x)\alpha(r)\alpha(y)= D(x)\alpha(y)\beta(x)+\beta(r)D(y)\beta(x)$.
Using the relation (23) we get $D(x)\alpha(r)\alpha(y)= D(r)\alpha(y) \beta(x)+\beta(r)D(x)\alpha(y)$.
Thus $D(r)\ \alpha(y)\beta(x)=[D(x)\alpha(r)-\beta(r)D(x)]\alpha(y)$.
Substituting $\alpha\ (r)$ by $r$and $\beta(r)$ by r in the above equation we get
$D(r)\ \alpha(y)\beta(x)= [D(x)r-rD(x)]\alpha(y)$.
Thus $D(r)\ \alpha(y)\beta(x)= [D(x),r]\alpha(y)$, (24)
for all $x,y,r\in U$.
We now replace r by $D(z),z\in U$. Then we find that
$D^2(z)\alpha(y)\beta(x)=[D(x),D(z)]\alpha(y)$, (25)
for all $x,y,z\in U$.
Now substituting $\alpha(y)=y$, $\beta(x)=x$, we obtain $D^2(z)yx=[D(x),D(z)]y$ and applying D to the condition $D(x)z=D(z)x$. That is $D^2(x)\alpha(z) + \beta D(xz)=D^2(z)\alpha(x)+\beta D(zx)$. Substituting xz by $\beta^{-1}xz$ and zx by $\beta^{-1}xz$, in the above equation we obtain $D^2(x)\alpha(z) + \beta D(\beta^{-1}xz)=D^2(z)\alpha(x)+ \beta D\beta^{-1}(zx)$, i.e., $D^2(x)\alpha(z)+[D(x),D(z)] = D^2(z)\alpha(x)$. Now multiplying with y on right $D^2(x)\alpha(z)y+[D(x),D(z)]y= D^2(z)\alpha(x)y$. Thus $[D(x),D(z)]\ y= D^2(z)\alpha(x)y-D^2(x)\alpha(z)y$.
We replace $\alpha(x)$ by x and $\alpha(z)$ by z to get
$[D(x),D(z)]\ y = D^2(z)xy-D^2(x)zy.$ (26)
Substituting in the relation (25), yields, $D^2(z)yx= [D(x),D(z)]y$ which implies $D^2(z)yx= D^2(z)xy-D^2(x)zy$, i.e.,$D^2(x)zy= D^2(z)[x,y]$, (27)
for all $x,y,z\in U$. Since $[x,y]$ is constant, applying D to the (22),we have that $D(U)[x,y] =\{0\}=D^2(U)[x,y]$, for all $x,y\in U$ and the relation (27) yields $D^2(U)U^2 =\{0\}$. Since $U^2\neq\{0\}$ and R is prime we can conclude that $D^2(U) = \{0\}$. Finally, since R is 2-torsion free, using the fact that $D^2(xy)=0$, for all $x,y\in U$ gives $D(U)D(U) = \{0\}$. Hence the Theorem is proved.♦

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