ON EQUITABLE COLORING OF PLICK AND LICT GRAPHS

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Abstract: The concept of "equitable coloring" was introduced by Meyer in the year 1973. In this paper, we discuss the equitable coloring of some graph valued functions/operations namely, plick graph and lict graph for certain class of graphs.

Keywords: Equitable coloring, lict graph, plick graph.

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Introduction: All graphs considered here are finite, undirected, connected, without loops and multiple edges. We follow the terminology of Harary[2]. For any graph G, V(G) and E(G) denotes the vertex set and edge set of graph G respectively. The order of G $|V(G)|$ is the number of vertices of G and the size of G $|E(G)|$ is the number of its edges.

If the set of vertices of a graph G can be partitioned into k-classes $V_1, V_2, V_3, \ldots, V_k$ such that each V_i is an independent set and the condition $\left||V_i| - |V_j|\right| \leq 1$ holds for all i ≠ j, then G is said to be *equitably k-colorable*, ($V_1, V_2, V_3, \ldots, V_k$) is called an *equitableindependence-partition.* The smallest integer k for which G is equitable k-colorable is known as the *equitable chromatic number* of a graph G [1,5,6,7] and is denoted by $\chi_e(G)$.

In the below figure 1, a graph G and its equitable coloring is shown

G:

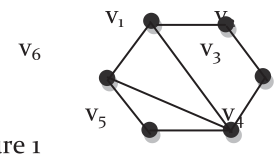

figure 1

Here independent sets are,

$V_1 = \{v_1, v_5\}$, $V_2 = \{v_2, v_4\}$, $V_3 = \{v_3, v_6\}$.

Clearly, $|V_1| = |V_2| = |V_3| = 2$ .Also, the condition $\left||V_i| - |V_j|\right| \leq 1$, for all i ≠ j, is satisfied.Therefore G is equitably 3-colorable.i.e., $\chi_e(G) = 3$.

For a real number x, $\lceil x \rceil$ denotes the smallest integer not less than x, and $\lfloor x \rfloor$ is the largest integer not greater than x.

The *Plick graph* P(G) [3] of a graph G is obtained from the line graph by adding a new point corresponding to each block of the original graph and joining this point to the points of the line graph which correspond to the lines of the block of the original graph.

Remark 1.1[3]: $V(P(G)) = E(G) \bigcup B_i$, where B_i ; i=1,2,3, . . . is the total number of blocks in graph G.

The cut-points and lines of a graph G are called its members. The *Lict graph* $\mathbf{L_C}$ (G) [4] of a graph has cut-points and lines of a graph G. The graph whose point set is the union of the lines and the set of cut-points of G in which two points are adjacent if and only if the corresponding lines of G are adjacent or the corresponding members of G are incident.

Remark 1.2[4]: $V(\mathbf{L_C}(G)) = E(G) \bigcup C_i$, where C_i ; i=1,2,3,4, . . . is the total number of cut-points in graph G.

2. Equitable Coloring of Plick Graph:

In this section, we obtain the equitable chromatic number of plick graph for certain class of graphs like star graph, cycle, path, complete bipartite graph and wheel.

Theorem 2.1 Equitable chromatic number of plick graph of star graph ($P(K_{(1,n)})$) is n.

Proof: Let $G = K_{(1,n)}$.

Then, by the definition of plick graph, V(P(G)) = 2n.

We consider the following cases depending on the vertex partition.

Case 1: Suppose, $V_1 = \{b_1, b_2, b_3, \ldots, b_n\}$,

$V_2 = \{e_1\}$, $V_3 = \{e_2\}$, $V_4 = \{e_3\}$, . . . , $V_n = \{e_{n-1}\}$, $V_{(n+1)} = \{e_n\}$.

Clearly $V_1, V_2, V_3, V_4, \ldots, V_n, V_{(n+1)}$ are independent sets.

$V(P(G)) = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4, \ldots, \bigcup V_n \bigcup V_{(n+1)}$ and $|V_1| = n$,

$|V_2| = |V_3| = |V_4| =, \ldots, = |V_{n+1}| = 1$.

Therefore the condition i.e., $\left||V_i| - |V_j|\right| \leq 1$ for all i ≠ j, is satisfied, only when $|V_1| = 1 or 2$, if $|V_1| > 2$ then, the condition fails.

Case 2: Let us consider the partition of sets in the following manner.

Now, let $V_1 = \{b_1, e_2\}$, $V_2 = \{b_2, e_3\}$, . . . , $V_{n-1} = \{b_{n-1}, e_n\}$ and $V_n = \{b_n, e_1\}$ be the least number of independence sets and

$V(P(G)) = V_1 \cup V_2 \cup V_3, \ldots, \cup V_{(n-1)} \cup V_n$.

Also, $|V_1| = |V_2| = |V_3| =, \ldots, = |V_n| = 2$.

Clearly, the condition i.e., $\left\| V_i \right| - \left| V_j \right\| \le 1$ for all i ≠ j, is satisfied.

Therefore the equitable chromatic number of plick graph of star graph P(G) is n.

i.e., χ_e (P(G)) = n.

Theorem 2.2: Equitable chromatic number of plick graph of cycle ($P(\mathbf{C_n})$) ; n ≥ 3 is

$$\chi_e(P(C_n)) = \begin{cases} 4 & if\ n = 3 \\ \frac{n}{2} + 1 & if\ n\ is\ even \\ \left\lceil \frac{n}{2} + 1 \right\rceil & if\ n\ is\ odd \end{cases}$$

Proof: Let P(G) = P(C_n) be the plick graph of cycle with n+1 vertices.

Now we partition the vertex set V(P(G)) as follows.

We consider the following two cases.

Case 1: If 'n' = 3.

Let $V_1 = \{b\}$, $V_2 = \{e_1\}$, $V_3 = \{e_2\}$, $V_4 = \{e_3\}$.

Here V_1, V_2, V_3 and V_4 are the independent sets and $V(P(G)) = V_1 \cup V_2 \cup V_3 \cup V_4$.

Thus, $|V_1| = |V_2| = |V_3| = |V_4| = 1$ and

$\left\| V_i \right| - \left| V_j \right\| = 0$.

Hence the condition i.e., $\left\| V_i \right| - \left| V_j \right\| \le 1$ for all i ≠ j, is satisfied.

Therefore, the equitable chromatic number of plick graph of cycle is 4 when n = 3.

i.e., χ_e (P(G)) = 4, when n = 3.

Case 2: When 'n' > 3.

In this case we consider the following two sub cases.

Subcase 2.1: If 'n' is even.

Let $V_1 = \{b\}$, $V_k = \{e_{k-1}, e_{\frac{n}{2}+k-1}\}$,

where 2 ≤ k ≤ ($\frac{n}{2} + 1$).

Here V_1 and V_k, where 2 ≤ k ≤ ($\frac{n}{2} + 1$)are the independent sets and $V(P(C_n)) = V_1 \cup V_k$ where 2 ≤ k ≤ ($\frac{n}{2} + 1$). Hence $|V_1| = 1$, $|V_k| = 2$, where 2 ≤ k ≤ ($\frac{n}{2} + 1$).

Thus the condition i.e., $\left\| V_i \right| - \left| V_j \right\| \le 1$ for all i ≠ j, is satisfied.

Hence the equitable chromatic number of plick graph of cycle is ($\frac{n}{2} + 1$), when n > 3 and even.

i.e., χ_e (P(G)) = $\frac{n}{2} + 1$, when n even.

Subcase 2.2: If 'n' is odd.

Let $V_1 = \{b\}$, $V_2 = \{e_1\}$ and $V_m = \{e_{m-1}, e_{\left\lfloor \frac{n}{2} \right\rfloor + m - l}\}$, where (3 ≤ m ≤ $\left\lceil \frac{n}{2} + 1 \right\rceil$) .

Here V_1, V_2 and V_m, where (3 ≤ m ≤ $\left\lceil \frac{n}{2} + 1 \right\rceil$) are the independent sets and $V(P(C_n)) = V_1 \cup V_2 \cup V_m$, where (3 ≤ m ≤ $\left\lceil \frac{n}{2} + 1 \right\rceil$). Hence $|V_1| = |V_2| = 1$ and $|V_m| = 2$, where (3 ≤ m ≤ $\left\lceil \frac{n}{2} + 1 \right\rceil$).

Thus the condition i.e., $\left\| V_i \right| - \left| V_j \right\| \le 1$ for all i ≠ j, is satisfied.

Hence the equitable chromatic number of plick graph of cycle P(G) is $\left\lceil \frac{n}{2} + 1 \right\rceil$ when n > 3 and even.

Hence χ_e (P (G)) = $\left\lceil \frac{n}{2} + 1 \right\rceil$, when 'n' odd.

$$\therefore \chi_e(P(G)) = \begin{cases} 4 & if\ n = 3 \\ \frac{n}{2} + 1 & if\ n\ is\ even \\ \left\lceil \frac{n}{2} + 1 \right\rceil & if\ n\ is\ odd \end{cases}$$

Theorem 2.3: Equitable chromatic number of plick graph of path is 2, when n ≥ 2.

Proof: Let G = P_n ; n ≥ 2.

By the definition of plick graph of graph G, V(P(G)) = 2n-2.

Partitioning the vertex set V(P(G)) depending on the following cases, we consider.

Case 1: If G = P_2 then P(G)= P_2.

Obviously, there exists only two independent sets i.e., $V_1 = \{b_1\}$ and

$V_2 = \{e_2\}$, and $V(P(G)) = V_1 \bigcup V_2$, also $|V_1| = |V_2| = 1$.

Thus the condition i.e., $\left||V_i| - |V_j|\right| \leq 1$ for all i ≠ j, is satisfied.
Hence the equitable chromatic number of plick graph of path is $\chi_e(P(G)) = 2$.
Case2: If $G = P_n$; n ≥ 3.
Suppose, $e_1 e_2 e_3 \ldots e_{n-1}$ be the path, $e_1, e_2, e_3, \ldots, e_{n-1}$ be the points corresponding to the edges of G and $b_1, b_2, b_3, \ldots, b_{n-1}$ be the points corresponding to the blocks of G and each b_i is adjacent to e_i; i=1,2,3,…,n-1. Also e_i and e_j are adjacent, 1≤ j ≤ n-1. Here 'n' is the number of vertices of G.
$V_1 = \{b_1, b_2, b_3, \ldots, b_{n-1}\}$, $V_2 = \{e_i$, even distance vertices from $e_i\}$
$V_3 = \{$odd distance vertices from $e_i\}$

Clearly, V_1, V_2 and V_3 are independent sets and $V(P(G)) = V_1 \bigcup V_2 \bigcup V_3$.
Further, if n > 3, then $|V_1| > |V_2|$ and $|V_3|$. Thus the condition i.e., $\left||V_i| - |V_j|\right| \leq 1$, for all i ≠ j, then the condition fails.
Now, let $V_1 = \{ b_i$; $i \equiv 1 \pmod 2$ where 1≤ i ≤ (n - 1)(if 'n' is even) or (n -2)(if 'n' is odd)$\} \bigcup \{ e_j$; $j \equiv 0 \pmod 2$ where 1 ≤ j ≤(n -1)(if 'n' is odd) or(n -2)(if 'n' is even) },
$V_2 = \{ b_j$; $j \equiv 0 \pmod 2$ where 1≤ j ≤(n - 1)(if 'n' is odd) or (n -2)(if 'n' is even)$\} \bigcup \{ e_i$; $i \equiv 1 \pmod 2$ where 1 ≤ i ≤(n - 1)(if 'n' is even) or (n -2)(if 'n' is odd) }.
Clearly, V_1 and V_2 are independent sets and $V(P(G)) = V_1 \bigcup V_2$.
Also, $|V_1| = |V_2| = n - 1$.

Hence the condition i.e., $\left||V_i| - |V_j|\right| \leq 1$ for all i ≠ j, is satisfied.
Therefore the equitable chromatic number of plick graph of path is 2.
i.e., $\chi_e(P(G)) = 2$.
Theorem 2.4: Equitable chromatic number of plick graph of complete bipartite graph $(P(K_{n,n}))$ is,

$$\chi_e(P(K_{n,n})) = \begin{cases} \frac{n^2}{2} + 1 & \text{if 'n' is even} \\ \left\lceil \frac{n^2}{2} + 1 \right\rceil & \text{if 'n' is odd} \end{cases}$$

Proof: Let $G = K_{n,n}$ be the complete bipartite graph with vertex set as
V(G) = 2n.
$= (v_1, v_2, v_3, \ldots, v_n \bigcup v_1', v_2', v_3', \ldots, v_n')$
$\because (n = n')$.
Let P(G) be the plick graph of complete bipartite graph with vertex set as,
$V(P(G)) = (v_1v_1', v_1v_2', v_1v_3', \ldots, v_1v_n') \bigcup$
$(v_2v_1', v_2v_2', v_2v_3', \ldots, v_2v_n') \bigcup$
$(v_3v_1', v_3v_2', v_3v_3', \ldots, v_3v_n') \bigcup, \ldots,$
$\bigcup (v_nv_1', v_nv_2', v_nv_3', \ldots, v_nv_n') \bigcup B_i$.
$V(P(G)) = (e_1, e_2, e_3, \ldots, e_n) \bigcup ($
$e_{n+1}, e_{n+2}, e_{n+3}, \ldots, e_{2n}) \bigcup, \ldots, \bigcup$
$(e_{[((n-1)n)+1]}, e_{[((n-2)n)+2]}, e_{[((n-1)n)+3]}, \ldots, e_{[((n-1)n)+n]}) \bigcup 1$.
$\because (B_i=1)$
$V(P(G)) = n^2+1$.
Now we partition the vertex set as follows.
Consider the following cases.
Case 1: If 'n' is even.
Let $V_1 = \{b\}$, $V_k = \{e_{k-1}, e_{n^2-k=2}\}$, where $\{2 \leq k \leq \frac{n^2}{2} + 1\}$.

Clearly, V_1 and V_k, where $\{2 \leq k \leq \frac{n^2}{2} + 1\}$ are independent sets and the elements in V_k, where $\{2 \leq k \leq \frac{n^2}{2} + 1\}$ are not repeated.

Also, $|V_1| = 1$ and $|V_k| = 2$, where $\{2 \leq k \leq \frac{n^2}{2} + 1\}$.

Hence the condition i.e., $\left||V_i| - |V_j|\right| \leq 1$ for all i≠j, is satisfied.
Therefore, the equitable chromatic number of plick graph of complete bipartite graph is $\frac{n^2}{2} + 1$.

Case2: If 'n' is odd.
In this case we have two sub cases.
Subcase2.1: If 'n' = 1.
Let $V_1 = \{b\}$ and $V_2 = \{e\}$.
Here V_1 and V_2 are the only two independent sets and $V(P(K_{1,1})) = V_1 \bigcup V_2$. Also, $|V_1| = |V_2| = 1$.

Hence the condition i.e., $\left||V_i| - |V_j|\right| \leq 1$, for all i≠j, is satisfied.

And the equitable chromatic number of plick graph of complete bipartite graph is 2 or $\frac{n^2}{2}+1$.

Subcase2.2: If 'n' >1.

Let $V_1=\{b\}$, $V_2=\{e_1\}$ and

$V_k=\{e_{k-1}, e_{n^2-k+3}\}$, where $\left(3 \le k \le \left\lceil \frac{n^2}{2}+1 \right\rceil\right)$.

Clearly, V_1, V_2 and V_k, where $\left(3 \le k \le \left\lceil \frac{n^2}{2}+1 \right\rceil\right)$ are independent sets with no repeats and $V(P(K_{n,n}))=V_1 \cup V_2 \cup V_k$, where $\left(3 \le k \le \left\lceil \frac{n^2}{2}+1 \right\rceil\right)$.

Hence the condition $\left||V_i|-|V_j|\right| \le 1$, for all $i \ne j$, is satisfied.

Therefore the equitable chromatic number of plick graph of complete bipartite graph i.e., $\chi_e(P(G)) = \left\lceil \frac{n^2}{2}+1 \right\rceil$ when n is odd.

$$\therefore \chi_e(P(G)) = \begin{cases} \frac{n^2}{2}+1 & \text{if 'n' is even} \\ \left\lceil \frac{n^2}{2}+1 \right\rceil & \text{if 'n' is odd} \end{cases}$$

Theorem 2.5: Equitable chromatic number of plick graph of wheel $P(W_n)$ is 'n' ; $n \ge 4$.

Proof: Let $G=W_n$ be the wheel ; $n \ge 4$.

Let P(G) be the plick graph of wheel. Then V(G) = $v_1, v_2, v_3, \ldots, v_n$ and

$V(P(G)) = (v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_1) \cup (v_nv_{n-1}, v_nv_1, v_nv_2, v_nv_3, \ldots, v_nv_{n-2})$

$= (e_1, e_2, e_3, \ldots, e_{n-1}) \cup (e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n-3}, e_{2n-2})$

Therefore now we partition the vertex set as follows

Let $V_1=\{b\}$, $V_i=\{e_{i-1}, e_{(n-1)+i-1}\}$, where $(2 \le i \le n)$.

Clearly, V_1 and V_i, where $(2 \le i \le n)$ are the independent sets and

$V(P(G))=(V_1 \cup V_i)$ where $(2 \le i \le n)$ .Then $|V_1|=1$ and $|V_i|=2$ where $(2 \le i \le n)$. Thus the condition i.e., $\left||V_i|-|V_j|\right| \le 1$ for all $i \ne j$, is satisfied. Hence the equitable chromatic number of plick graph of wheel (P(G)) = n.

i.e., $\chi_e(P(G)) = n$.

3. Equitable Chromatic Number Of Lict Graph For Star Graph And Path:

Theorem 3.1: Equitable chromatic number of lict graph of star graph is 'n+1'.

Proof: Let $G = K_{1,n}$

Let $L_c(G)$ be the lict graph of graph G,

$V(L_c(G))=n+1$.

Depending on the vertex set we consider,

$V_1=\{c\}, V_2=\{e_1\}, V_3=\{e_2\}, \ldots, V_{n+1}=\{e_{n-1}\}$

Clearly, $V_1, V_2, V_3, \ldots, V_n, V_{n+1}$ are independent sets and

$V(L_C(K_{1,n}))= V_1 \cup V_2 \cup V_3 \cup \ldots, V_n \cup V_{n+1}$.

Also, $|V_1|=|V_2|=|V_3|=,\ldots,=|V_n|=|V_{n+1}|=1$.

Here the condition $\left||V_i|-|V_j|\right| \le 1$ for all $i \ne j$, is satisfied.

Therefore equitable chromatic number of lict graph of star graph is n+1.

i.e., $\chi_e(L_C(G)) = n+1$.

Theorem 3.2: Equitable chromatic number of lict graph of path is 3; $n \ge 3$.

Proof: Let $G = P_2$, then the case is trivial.

Let $G = P_n$, $n \ge 3$ by the definition of lict graph of graph G,

$V(L_C(G))= \{e_1, e_2, e_3, \ldots, e_{n-1}\} \cup \{c_1, c_2, c_3, \ldots, c_{n-2}\}$

Considering the partitioning of the vertex set as follows we have,

$V_1 = \{e_i;\ i \equiv 1 \pmod 3$ where $1 \le i \le (n-1)\} \cup \{c_j;\ j \equiv 2 \pmod 3$ where $2 \le j \le (n-2)\}$,

$V_2 = \{e_i;\ i \equiv 2 \pmod 3$ where $2 \le i \le (n-1)\} \cup \{c_j;\ j \equiv 0 \pmod 3$ where $3 \le j \le (n-2)\}$,

$V_3 = \{e_i;\ i \equiv 0 \pmod 3$ where $3 \le i \le (n-1)\} \cup \{c_j;\ j \equiv 1 \pmod 3$ where $1 \le j \le (n-2)\}$.

Clearly, V_1, V_2 and V_3 are independent sets with no repetitions and

$V(L_C(P_n)) = V_1 \cup V_2 \cup V_3$.

Then $|V_1|=|V_2|=|V_3|=\left\lceil \frac{2n-3}{3} \right\rceil$ when $n \in 3k, k=1,2,3,\ldots$

$|V_1|=\left\lceil \frac{2n-3}{3} \right\rceil$ and $|V_2|=|V_3|=\left\lfloor \frac{2n-3}{3} \right\rfloor$ where $n \in 3k-1, k=1,2,3,\ldots$

$\left|V_1\right| = \left|V_2\right| = \left\lceil \dfrac{2n-3}{3} \right\rceil$ and $\left|V_2\right| = \left\lfloor \dfrac{2n-3}{3} \right\rfloor$ where $n \in 3k-2, k = 1,2,3, \ldots$

Here the condition $\left\|V_i\right| - \left|V_j\right\| \le 1$ for all i≠ j, is satisfied.

Therefore equitable chromatic number of lict graph of path is 3.

i.e., $\chi_e(\mathbf{L_C}(G)) = 3$.

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