ON COVERING ENERGY OF POSETS

M. M. PAWAR, VANDANA P. BHAMARE

Abstract: The aim of this paper is to introduce the concept energy of a poset. For energy of a poset upper and lower bounds are established. Energies of some well known posets are computed and the counter example is provided to settle the conjecture by Bapat and Pati.

Keywords: poset, chain, lattice, vertices, edges, covering matrix, characteristic polynomial, eigenvalue, covering energy.

Introduction and Preliminaries: During 1930s E. Huckel[9] put forward a method for finding approximate solution of the Schrodinger equation of a class of organic molecules, nowadays known as Huckel Molecular Orbital [HMO] theory. In this theory the total energy of π-electrons is equal to the sum of the energies of π-electrons in the considered molecule. It can be calculated from the eigenvalues of the underlying molecular graph. Motivated by HMO total π-electron energy I. Gutman[7] introduced the concept of *energy of a graph* as sum of the absolute values of all eigenvalues of incidence matrix of a graph. He opened a door for new research area in Mathematics and Chemistry. Energy likes quantities were considered also for matrices: Laplacian[8], Distance[10], Minimum covering[1], etc.. In the literature there are many papers which interlink graph theory and lattice theory. In this paper we introduce *covering matrix* of a poset, and study its eigenvalues and energy.

Here we recall some basic definitions. A non empty set P, together with a binary relation ≤ which is reflexive, antisymmetric and transitive is called a partially order set in short Poset. A Hasse diagram is a type of mathematical diagram used to represent a finite partially ordered set, in the form of a drawing of its transitive reduction. Concretely, for a partially ordered set $(P,\leq)$ one represents each element of P as a vertex in the plane and draw a line segment or curve that goes upward from x to y whenever y covers x(i.e., whenever $x<y$and there is no z such that$x<z<y$, denoted as $x \prec y$). These curves may cross each other but must not touch any vertex other than their endpoints; we call such a curve as an *edge*. An element x in a poset $P$is *doubly-irreducible* if it covers and covered by at most one element. The concept of doubly irreducible element was introduce by I. Rival[14], for more details see [5][15]. The set of all doubly irreducible elements in P is denoted by$Irr(P)$. An element in P which is not doubly irreducible is called *reducible* and *the* set of all reducible elements in P is denoted by$R(P)$. Elements a,b are said to be *comparable* if either $a \leq b$or $b \leq a$otherwise they are said to be *incomparable.* A poset in which every pair of elements is comparable, is called a *chain* and if every pair of elements is incomparable, is called an *antichain.* A *lattice* is a poset in which every pair of elements have a supremum (the least upper bound; called their join) and an infimum (the greatest lower bound; called their meet). If *a, b* are elements in a lattice *L* then their join and meet are denoted by $a \vee b$ and $a \wedge b$ respectively. For undefined terms see Gratzer[6], Davey and Priestley[4].

Following lemma can be found in Bapat[2], it is used very frequently in the study of energy of graphs.

Lemma 1.1: Let M be a non-singular matrix, and let *N, P* and Q be matrices such that $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$be a square matrix then $|S| = |M||Q-PM^{-1}N|$.

2. Definition and Bounds: We introduce a concept of covering energy of a poset.

Definition 2.1: Let *(P, ≤)* be a poset of order n with $P=\{v_1,v_2,v_3,\ldots.v_n\}$ then the *covering matrix* of P is $n \times n$ matrix defined by *A(P)*=(a_{ij}), where

$$a_{i,j} = \begin{cases} 1 & if v_i \prec v_j or v_j \prec v_i \\ 1 & if i = j and v_i \notin Irr(P) \\ 0 & otherwise \end{cases}$$

The characteristic polynomial $\phi(P,\lambda)$ of a poset P with n elements is the determinant $|A(P)-\lambda I_n|$. The eigenvalues of *A(P)* are called *eigenvalues* of the *poset P* and the sum of absolute values of eigenvalues of the poset P is called *covering energy* of the poset P denoted by *E(P).*

As *A(P)* is real and symmetric all its eigenvalues are real numbers, we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, called *spectrum* of the poset *P.* Clearly,

E(P) = $\sum_{i=1}^{n} |\lambda_i|$.

Hence onwards by energy of poset we mean covering energy of poset and whenever we are dealing with only one poset we will denote $|R(P)|$ by $|R|$.

It is observed that the energy of a poset is unique up to isomorphism. Two non-isomorphic posets may have the same energy. From the definition it is cleared that dual poset have the same energy. We compute energy of two simple posets.

Example 2.2: Let M_2 be a four element lattice as depicted in Figure 1 then

$$A(M_2) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

The characteristic polynomial of M_2 is $\phi(M_2, \lambda) = 4\lambda - 3\lambda^2 - 2\lambda^3 + \lambda^4$,

Eigenvalues are $\{\frac{1+\sqrt{17}}{2}, 1, 0, \frac{1-\sqrt{17}}{2}\}$ and $E(M_2) = 1 + \sqrt{17}$.

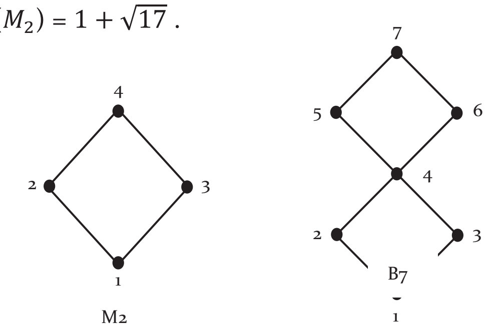

Figure 1

Example 2.3: Let B_7 be a seven element lattice as depicted in Figure 1 then

$$A(B_7) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The characteristic polynomial of B_7 is

$\phi(B_7, \lambda)$

$= -12\lambda^2 + 4\lambda^3 + 15\lambda^4 - 5\lambda^5 - 3\lambda^6 + \lambda^7$,

Eigenvalues are $\{3, 2, 1, 0, 0, -1, -2\}$ and $E(B_7) = 9$.

Theorem2.4: Let P be a poset of order n and $\phi(P,\lambda) = \sum_{i=1}^{n} a_i \lambda^i$ be the characteristic polynomial of P then

(1) $a_n = 1$,

(2) $a_{n-1} = |Irr(P)| - n$,

(3) $a_{n-2} = \binom{|R|}{2} - |E|$,

(4) $a_{n-3} = |R||E| - \sum_{v \notin Irr(P)} d(v) - \binom{|R|}{3}$.

Proof: (1) From the definition of $\phi(P,\lambda)$ it follows that $a_n = 1$.

(2) Since $(-1)^i a_{n-i}$ = sum of the principal minors of $A(P)$ of order i.

$a_{n-1} = (-1)^1$trace of $A(P)$

$= -|R|$

$= |Irr(P)| - n.$

(3) The sum of principal minors of order 2 of the matrix $A(P) = (-1)^2 a_{n-2}$. Hence,

$$a_{n-2} = \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

$$= \sum_{1 \le i < j \le n} a_{ii}\, a_{jj} - \sum_{1 \le i < j \le n} a_{ij}\, a_{ji}$$

$$= \sum_{1 \le i < j \le n} a_{ii}\, a_{jj} - \sum_{1 \le i < j \le n} a_{ij}$$

$$= \binom{|R|}{2} - |E|.$$

(4) As the sum of principal minors of order 3 of the matrix A$(P) = (-1)^3 a_{n-3}$. Hence,

$$a_{n-3} = (-1)^3 \sum_{1 \le i < j < k \le n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}$$

$$= -\sum_{1 \le i < j < k \le n} (a_{ii}(a_{jj}a_{kk} - a_{kj}a_{jk}) - a_{ij}(a_{ji}a_{kk} - a_{ki}a_{jk}) + a_{ik}(a_{ji}a_{kj} - a_{jj}a_{ki}))$$

$$= -\sum_{1 \le i < j < k \le n} (a_{ii}a_{jj}a_{kk}) - 2\sum_{1 \le i < j < k \le n} (a_{ij}a_{jk}a_{ki}) + \sum_{1 \le i < j < k \le n} (a_{ii}a_{kj} + a_{jj}a_{ki} + a_{kk}a_{ij})$$

Since, there is no triangle in a poset.

$$= -\binom{|R|}{3} + \sum_{1 \le i < j < k \le n} (a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}),$$

$$= \left(\sum_{1 \le i < j < k \le n} a_{ii}\right)\left(\sum_{1 \le i < j < k \le n} a_{jk}\right) - \left(\sum_{i=1}^{n} a_{ii}\right)\left(\sum_{k=1, k \ne i}^{n} a_{ik}\right) - \binom{|R|}{3},$$

$= |R||E| - \sum_{v \notin Irr(P)} d(v) - \binom{|R|}{3}$. This completes the proof. ■

Corollary 2.5: If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are eigenvalues of a poset P then

(1) $\sum_{i=1}^{n} \lambda_i = n - |Irr(P)|$.

(2) $\sum_{i=1}^{n} {\lambda_i}^2 = 2|E| + n - |Irr(P)|$.

Proof: As $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are roots of the equation $\phi(P,\lambda) = \sum_{i=1}^{n} a_i \lambda^i = 0$, We have $\sum_{i=1}^{n} \lambda_i = -a_{n-1}$ and $\sum_{i=1}^{n} {\lambda_i}^2 = (a_{n-1})^2 - 2\, a_{n-2}$. Using Theorem 2.4 the result follows. ■

Using Corollary 2.5 we can obtain lower bound and upper bound for energy of a poset as follows.

Theorem 2.6: *If a poset P has n elements, m edges and* $D = |detA(P)|$ *then*

$$\sqrt{2m + n - |Irr(P)| + n(n-1)D^{\frac{2}{n}}} \le E(P) \le \sqrt{n(2m + n - |Irr(P)|)}.$$

Proof: Let $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \ldots \ge \lambda_n$ be the eigenvalues of a poset P. By using Cauchys -Schwarz inequality,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

with $a_i = 1$ and $b_i = |\lambda_i|$

and Corollary 2.5 we have

$$E(P)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2$$

$$\le n\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$= n\sum_{i=1}^{n} \lambda_i^2$$

$$= n(2m + n - |Irr(P)|)$$

i.e. $E(P) \leq \sqrt{n(2m+n-|Irr(P)|)}$ -----(A)

Also, consider

$$|E(P)|^2 = (\sum_{i=1}^{n} |\lambda_i|)^2$$
$$= (\sum_{i=1}^{n} |\lambda_i|)(\sum_{i=1}^{n} |\lambda_j|)$$
$$= (\sum_{i=1}^{n} |\lambda_i|^2) + \sum_{i\neq j} |\lambda_i|\,|\lambda_j|$$

Employing the inequality between the arithmetic and geometric means,

$$\frac{1}{n(n-1)} \sum_{i\neq j} |\lambda_i|\,|\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i|\,|\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

We have,

$$|E(P)|^2$$
$$\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i|\,|\lambda_j|\right)^{\frac{1}{n(n-1)}}$$
$$\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$
$$= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|\right)^{\frac{2}{n}}$$
$$= 2m + n - |Irr(P)| + n(n-1)D^{\frac{2}{n}}$$

Thus,

$$\sqrt{2m + n - |Irr(P)| + n(n-1)D^{\frac{2}{n}}} \leq E(P).$$

This inequality along with inequality (A) proves the result. ■

Bapat and Pati [3] proved that energy of a graph is never an odd integer. In the same paper they claim that, *If A is $n \times n$ symmetric matrix with integral entries then the sum of moduli of the eigenvalues of A cannot be an odd integer*. In their personal communication Professor S. B. Rao also conjectured the same.

Remark 2. 7: The matrix $A(B_7)$ used in Example 2.3 is symmetric with integral entries and provide a counter example to disprove the above conjecture. In fact, while establishing the result on energy of graphs they have prominently used the fact that trace of A is zero. But trace of the matrix $A(B7)$ is non zero. Therefore unlike graphs, the lattice B_7 also provides us an example of a poset whose covering energy is an odd integer.

Analogous to the result of Bapat and pati[3], Adiga et al[1, Theorem 3.7] established a result (Parity Theorem) for minimum covering energy of graphs, mutatis mutandis we extend it to posets as follows.

Theorem 2. 8: If energy $E(P)$ of a poset P is a rational number, then $E(P) \equiv |R| (mod 2)$

3. COVERING ENERGY OF SOME FAMILIES OF POSETS

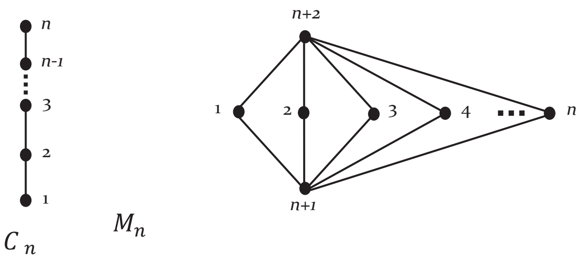

Figure 2

Let us denote a chain with n elements by C_n. The digram of C_n is as depicted in Figure 2 and the covering matrix is as follows.

$$A(C_n) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

It is observed that above covering matrix $A(C_n)$ and the adjacency matrix of a path of length n, $A(P_n)$are same. Hence we have:

Theorem 3.1: For$n \geq 2$, energy of a chain $C_n$is

$$= \begin{cases} 2Cosec\left(\frac{\pi}{2(n+1)}\right) - 2, ifniseven \\ 2Cot\left(\frac{\pi}{2(n+1)}\right) - 2, ifnisodd \end{cases}$$

For more details see [12, page 26]. Now we discuss energy of a lattice $M_n$on $n+2$ elements as depicted in Fig. 2.

Theorem 3.2: For a lattice M_n , n≥ 2

(1) $E(M_n) = 1 + \sqrt{8n+1}$,

(2) $E(M_n)$ can not be odd integer.

(3) $E(M_n)$ is an even integer when n is of the type $r(2r-1) or r(2r+1)$.

Proof: The covering matrix of $M_n$is as follows.

$$A(M_n) = \begin{bmatrix} 0 & 0 & 00 & \dots 0 & 1 & 1 \\ 0 & 0 & 00 & \dots 0 & 1 & 1 \\ \vdots & \vdots & \vdots\vdots & \dots\vdots & \vdots & \vdots \\ 0 & 0 & 00 & \dots 0 & 1 & 1 \\ 1 & 1 & 11 & \dots 1 & 1 & 0 \\ 1 & 1 & 11 & \dots 1 & 0 & 1 \end{bmatrix}$$

Thus

$$\emptyset(M_n, \lambda) = \begin{vmatrix} \lambda & 0 & 0 & & 0 & -1 & -1 \\ 0 & \lambda & 0 & \cdots & 0 & -1 & -1 \\ 0 & 0 & \lambda & & 0 & -1 & -1 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & & \lambda & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda-1 & 0 \\ -1 & -1 & -1 & & -1 & 0 & \lambda-1 \end{vmatrix}$$

Using Lemma 1.1 we have

$$\emptyset(M_n, \lambda) = \lambda^n \left| \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix} - Q(\lambda I_n)^{-1} Q^t \right|$$

where Q is 2 xn matrix $\begin{bmatrix} -1 & -1\ldots & -1 \\ -1 & -1\ldots & -1 \end{bmatrix}$

After simple calculations we obtain

$$\emptyset(M_n, \lambda) = \lambda^{n-1}(\lambda - 1)(\lambda^2 - \lambda - 2n).$$

which leads to

$$Spec(M_n) = \begin{pmatrix} 1 & n-1 & 1 & 1 \\ \frac{1-\sqrt{1+8n}}{2} & 0 & 1 & \frac{1+\sqrt{1+8n}}{2} \end{pmatrix}$$

and $E(M_n) = 1 + \sqrt{1+8n}$.

(2) If $E(M_n)$ is irrational then we have nothing to prove. Otherwise by Theorem 2.8 it must be integer say k then in the light of (1) we have,$(k-1)^2 = 1 + 8n$implies

$8|k(k-2)$and hence $E(M_n)$ must be even integer.

(3)If $E(M_n)$ is an even integer then $8n+1 = (2k-1)^2$ for some k. This implies $2n = k(k-1)$. $i.e.$either$k$ or $k-1$is even, and the result follows.

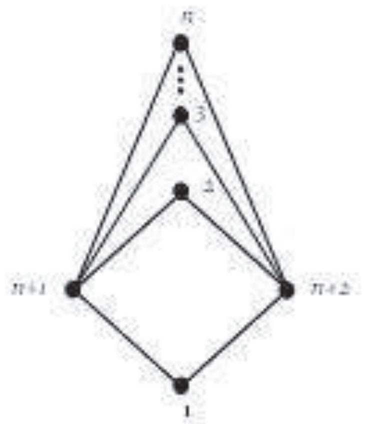

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Figure 3. M_n^T

In the poset M_n we observe that there are n incomparable elements, all of them are doubly irreducible and it is a lattice having two reducible elements. The poset M_n^Tas depicted in Figure 3 is not a lattice. Although M_n and M_n^T look isomorphic as graphs, they are not isomorphic as posets. In fact M_n^T contains no doubly irreducible element. We compute the covering energy of this poset.

Theorem 3.3: For a poset M_n^T , n≥ 2

(1) $E(M_n^T) = n + 2\sqrt{2n}$,

(2) $E(M_n^T)$ can not be odd integer.

(3) $E(M_n^T)$ is an even integer when n is of the type $2r^2$.

Proof: The covering matrix of $M_n^T$is as follows :

$$A(M_n^T) = \begin{bmatrix} 1 & 0 & 00 & \ldots 0 & 1 & 1 \\ 0 & 1 & 00 & \ldots 0 & 1 & 1 \\ \vdots & \vdots & \vdots\vdots & \ldots\vdots & \vdots & \vdots \\ 0 & 0 & 00 & \ldots 1 & 1 & 1 \\ 1 & 1 & 11 & \ldots 1 & 0 & 1 \\ 1 & 1 & 11 & \ldots 1 & 1 & 0 \end{bmatrix}$$

Using the same technique as in Theorem 3.2, we obtain

$$\emptyset(M_n^T, \lambda) = (\lambda - 1)^n(\lambda^2 - 2\lambda - 2n + 1).$$

This leads to

$$Spec(M_n^T) = \begin{pmatrix} 1 & n & 1 \\ 1-\sqrt{2n} & 1 & 1+\sqrt{2n} \end{pmatrix}$$

and $E(M_n^T) = n + 2\sqrt{2n}$. This proves (1) and easily leads to (2) and (3). ■

Ramane et al[13] studied equienergetic graph. On the same lines, if two non-isomorphic poset have the same energy they are said to be equienergetic. Changing the comparabilities of elements, we can construct *n+1* posets equienergetic to $E(M_n^T)$.

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M. M. Pawar/Principal/T.E.S`s Smt. V.U.Patil Arts and Late Dr. B.S.Desale Science College/
SAKRI Dist./Dhule (M.S.)/ 424-304/ INDIA.
Vandana P. Bhamare/Department of Applied Science/ Shri. Gulabrao Deokar College of Engg./
JALGAON (M.S.)/ 425-002/ INDIA.