ZERO-DIVISOR GRAPHS OF SOME SPECIAL LATTICES

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Abstract: The concept of the zero-divisor graphs for L_n,$L_n^{1^2}$ lattices are studied. In such graphs, the set of neighbourhoods of atom and dual atom gives equal number of elements. We have shown that the cardinality of neighbourhood of an element in zero-divisor graph is given by binomial expansion.

Keywords: Atom, dual atom, neighbourhood of an element, zero-divisor.

Introduction: The study of graphs associated with rings was initiated by Beck [3]. Two elements x, y in a commutative ring R are called adjacent if and only if $xy = 0$. He studied coloring of such graphs. Anderson and Livingston [2] studied graphs on rings by defining the adjacency of two non-zero elements $x, y \in R$ by $xy = 0$ and called these graphs as the zero divisor graph of R. Filipov [5] discuss the comparability graphs of partially ordered sets by defining the adjacency between two elements of a poset by using the comparability relation, that is a, b are adjacent if either $a \leq b$ or $b \leq a$. Duffus and Rival [1] discuss the properties of covering graphs derived from lattices. Nimbhorkar, Wasadikar and Pawar [6] defined the zero-divisor graph of lattice L with 0, by defining the adjacency of two elements $x, y \in L$ by $x \wedge y = 0$. Recently, Wasadikar and Survase [4] discussed incomparability graphs of lattices. If two elements a, b in a lattice are incomparable, we write $a||b$. Let L be a finite lattice and $W(L) = \{ x \in L|$ there exists $y \in L$ such that $x||y \}$. The incomparability graph of a lattice L, is a graph with the vertex set $W(L)$ and two distinct vertices $a, b \in W(L)$ are adjacent if and only if they are incomparable.

L_n is the lattice of divisors of a positive integer n of the form $n = p_1 \times p_2 \times p_3 \times \cdots \times p_k$.

$L_n^{1^2}$ is the lattice of divisors of a positive integer n of the form $n = p_1^2 \times p_2 \times p_3 \times \cdots \times p_k$.

In this paper, we discuss zero-divisor graph of L_n, $L_n^{1^2}$ lattices. For any positive integer n, the set of all divisors of n form a lattice with respect to divisibility relation. For $a, b \in L_n, L_n^{1^2}$, $a \leq b$ if $a|b$ or $b \leq a$ if $b|a$. The zero-divisor graph of lattice L_n, $L_n^{1^2}$ is denoted by $\Gamma(L_n)$ and $\Gamma(L_n^{1^2})$ respectively. An element $a \in L$ is called a zero-divisor if there exists a non-zero $b \in L$ such that $a \wedge b = 0$. We denote by $N(p)$ the set of all elements adjacent to p.

$N(p)$ neighbourhood of vertex p in graph is the induced sub graph of G consisting of all vertices adjacent to p and all edges connecting two such vertices. A graph is a representation of a set of objects where some pairs of objects are connected by links. An element x of L is an atom if $0 < x$ and there exists no element y of L such that $0 < y < x$.

Main Results: We show that cardinality of $N(p)$ is given by using binomial expansion. Initially we prove some results for vertices in $\Gamma(L_n)$, $\Gamma(L_n^{1^2})$ which are atoms.

Theorem 1: If L_n be a lattice of all divisors of a positive integer n with respect to divisibility of the form $n = p_1 \times p_2 \times p_3 \cdots \times p_k$ (all p_k are distinct) then in the zero-divisor graph of lattice L_n is,

$$|N(p_i)| = \sum_{r=1}^{k-1} {}^{k-1}c_r, \quad i = 1,2,3, \cdots k.$$

Proof: Let L_n be a lattice of all divisors of positive integer n with respect to divisibility where $p_1, p_2, p_3, \ldots, p_k$ all are prime factors of n and $\Gamma(L_n)$ is the zero-divisor graph of lattice L_n. For $a, b \in \Gamma(L_n)$ if $a \nmid borb \nmid a$ such that $a \wedge b = 0$ then $a - b$ is an edge in graph.

Let $p_i \in L_n$ then

$$p_i \nmid p_j\ ,\ \ p_i \wedge p_j = 0, \qquad i \neq j, \qquad i,j = 1,2,3, \ldots k,$$

i.e. $p_i - p_j$ is an edge in graph, so p_i has all $\{p_j,\ j = 1,2,3, \cdots, k\}$ neighbourhoods and these neighbourhoods are counting in ${}^{k-1}c_1$ ways.

Now taking product of two distinct elements from $\{p_j,\ j = 1,2,3, \cdots k\}$,

$p_i \nmid (p_{j_1} \times p_{j_2}), p_i \wedge (p_{j_1} \times p_{j_2}) = 0$,
$i \neq j,\ \ i,j = 1,2,3, \cdots k,$

i.e. $p_i - (p_{j_1} \times p_{j_2})$ is an edge in graph, so p_i has all $\{p_{j_1} \times p_{j_2}\}$ neighbourhoods and these neighbourhoods are counting in ${}^{k-1}c_3$ ways.

Similarly taking product of three elements from $\{p_j,\ j = 1,2,3, \cdots k\}$,

$p_i \nmid (p_{j_1} \times p_{j_2} \times p_{j_3}),\ \ p_i \wedge (p_{j_1} \times p_{j_2} \times p_{j_3}) = 0,\ i \neq j,$
$i,j = 1,2,3, \cdots k,$

i.e. $p_i - (p_{j_1} \times p_{j_2} \times p_{j_3})$ is an edge in graph, so p_i has all $\{p_{j_1} \times p_{j_2} \times p_{j_3}\}$ as neighbourhoods and these neighbourhoods are all counting in ${}^{k-1}c_3$ ways.

Continuing like this for product of $(k-1)$ elements from $\{p_j,\ j = 1,23, \cdots k\}$,

$$p_i \nmid (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}),$$

$p_i \wedge (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}) = 0,$
$i \neq j,\ \ i,j = 1,2,3, \cdots k,\ r = 1,2, \cdots k-1,$

i.e. $p_i - (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r})$ is an edge in graph, so p_i has $\{p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}\}$ as neighbourhoods and they are counting in ${}^{k-1}c_{k-1}$ ways.

Therefore neighbourhoods of p_i are,

$$^{k-1}c_1 + {}^{k-1}c_2 + {}^{k-1}c_3 + \cdots + {}^{k-1}c_{k-1}$$

$$i.e \sum_{r=1}^{k-1} {}^{k-1}c_r .$$

Hence,

$$|N(p_i)| = \sum_{r=1}^{k-1} {}^{k-1}c_r , i = 1,2,3,\cdots k.$$

Theorem 2: If $L_n^{1^2}$ be a lattice of all divisors of a positive integer n with respect to divisibility of the form $n = p_1^2 \times p_2 \times p_3 \times \cdots \times p_k$ (all p_k are distinct) then in the zero-divisor graph of lattice $L_n^{1^2}$ is,

$$|N(p_1)| = \sum_{r=1}^{k-1} {}^{k-1}c_r .$$

Proof: Let $L_n^{1^2}$ be a lattice of all divisors of positive integer n with respect to divisibility where $p_1, p_2, p_3, \cdots p_k$ all are prime factors of n and$\Gamma(L_n^{1^2})$ is the zero-divisor graph of lattice $L_n^{1^2}$. For $a, b \in \Gamma(L_n^{1^2})$ if $a \nmid b or b \nmid a$ such that $a \wedge b = 0$ then $a - b$ is an edge in graph. Let $p_1 \in L_n^{1^2}$ then

$p_1 \nmid p_j , p_1 \wedge p_j = 0, \ j = 2,3,4, \cdots k,$

i.e. $p_1 - p_j$ is an edge in graph, so p_1 has all $\{p_j, \ j = 2,3,4, \cdots k\}$ neighbourhoods and these neighbourhoods are counting in ${}^{k-1}c_1$ ways.

Now taking product of two distinct elements from$\{p_j, \ j = 2,3,4, \cdots k\}$then

$$p_1 \nmid (p_{j_1} \times p_{j_2}) ,$$

$p_1 \wedge (p_{j_1} \times p_{j_2}) = 0, \ j = 2,3,4, \cdots k,$

i.e. $p_1 - (p_{j_1} \times p_{j_2})$ is an edge in graph, so p_1 has all $\{p_{j_1} \times p_{j_2}\}$ as neighbourhoods and these neighbourhoods are counting in ${}^{k-1}c_2$ ways.

Continuing like this for product of $(k-2)$ elements from $\{p_j, \ j = 2,3,4, \cdots k\}$,

$$p_1 \nmid (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}),$$

$p_1 \wedge (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}) = 0,$

$j = 2,3,4, \cdots k, \ r = 1,2,3, \cdots k-2,$

i.e. $p_1 - (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r})$ is an edge in graph, so p_1 has $\{p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}\}$ as neighbourhoods and are counting in ${}^{k-1}c_{k-2}$ ways.

Lastly, $p_1 \nmid (p_2 \times p_3 \times p_4 \times \cdots \times p_k)$,

$p_1 \wedge (p_2 \times p_3 \times \cdots \times p_k) = 0,$

i.e. $p_1 - (p_2 \times p_3 \times \cdots \times p_k)$ is an edge in graph, so p_1 has $\{p_2 \times p_3 \times \cdots \times p_k\}$ is one neighbourhood counting in ${}^{k-1}c_{k-1}$ way.

Therefore neighbourhoods of p_1 are,

$$^{k-1}c_1 + {}^{k-1}c_2 + \cdots + {}^{k-1}c_{k-2} + {}^{k-1}c_{k-1}$$

$$i.e. \sum_{r=1}^{k-1} {}^{k-1}c_r .$$

Hence,

$$|N(p_1)| = \sum_{r=1}^{k-1} {}^{k-1}c_r .$$

Theorem 3: If $L_n^{1^2}$ be a lattice of all divisors of a positive integer n with respect to divisibility of the form $n = p_1^2 \times p_2 \times p_3 \times \cdots \times p_k$ (all p_k are distinct) then in the zero-divisor graph of lattice $L_n^{1^2}$ is,

$$|N(p_i)| = \sum_{r=1}^{k-1} {}^{k-1}c_r + \sum_{r=1}^{k-2} {}^{k-2}c_r + 1,$$

$$i = 2,3,4, \cdots k.$$

Proof: Let $L_n^{1^2}$ be a lattice of all divisors of positive integer n with respect to divisibility where $p_1, p_2, p_3, \cdots p_k$ all are prime factors of n and $\Gamma(L_n^{1^2})$ is the zero-divisor graph of lattice $L_n^{1^2}$. For $a, b \in \Gamma(L_n^{1^2})$ if $a \nmid b or b \nmid a$ such that $a \wedge b = 0$ then $a - b$ is an edge in graph.

Let $p_i \in L_n^{1^2}$ then

$$p_i \nmid p_j , \ p_i \wedge p_j = 0,$$

$$i \neq j, \ i = 2,3,4, \cdots k, \ j = 1,2,3, \cdots k,$$

i.e. $p_i - p_j$ is an edge in graph, so p_i has all $\{p_j, \ j = 1,2,3, \cdots k\}$ neighbourhoods and these neighbourhoods are all counting in ${}^{k-1}c_1$ ways.

Now we take product of two elements from $\{p_j, \ j = 1,2,3, \cdots k\}$ then

$$p_i \nmid (p_{j_1} \times p_{j_2}) , p_i \wedge (p_{j_1} \times p_{j_2}) = 0,$$

$i \neq j, \ i = 2,3,4, \cdots k, \ j = 1,2,3, \cdots k,$ i.e.

$p_i - (p_{j_1} \times p_{j_2})$ is an edge in graph, if $p_{j_1} \neq p_{j_2}$ then $\{p_{j_1} \times p_{j_2}\}$are counting in ${}^{k-1}c_2$ ways,

else $p_{j_1} = p_{j_2}$ i.e. $\{p_1 \times p_1\}$ counting in ${}^{k-2}c_{k-2}$ way.

Similarly taking product of three elements from $\{p_j, \ j = 1,2,3, \cdots k\}$, $p_i \nmid (p_{j_1} \times p_{j_2} \times p_{j_3})$,

$p_i \wedge (p_{j_1} \times p_{j_2} \times p_{j_3}) = 0,$

$i \neq j, \ i = 2,3,4, \cdots k, \ j = 1,2,3, \cdots k,$

i.e. $p_i - (p_{j_1} \times p_{j_2} \times p_{j_3})$ is an edge in graph.

If $p_{j_1}, p_{j_2}, p_{j_3}$ are distinct then p_i has all $\{p_{j_1} \times p_{j_2} \times p_{j_3}\}$ as neighbourhoods and these neighbourhoods are all counting in ${}^{k-1}c_3$ ways,

else are counting in ${}^{k-2}c_{k-3}$ ways.

Continuing like this for product of $(k-1)$ elements from $\{p_j, \ j = 1,2,3, \cdots k\}$,

$$p_i \nmid (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}) ,$$

$$p_i \wedge (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}) = 0, \ i \neq j,$$

$$i = 2,3, \cdots k, \ j = 1,2, \cdots k,$$

$$r = 1,2, \cdots k-1.$$

i.e. $p_i - (p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r})$ is an edge in graph.

If all p_{j_r} are distinct then they are counting in ${}^{k-1}c_{k-1}$ way,

else are counting in ${}^{k-2}c_1$ ways.

Last $p_i \nmid (p_1 \times p_1 \times p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r})$,

$$p_i \wedge (p_1 \times p_1 \times p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}) = 0 ,$$

$$i \neq j, \ i, j = 2,3, \cdots k, \ r = 1,2, \cdots k-2,$$

which is one neighbourhood of p_i i.e. $p_i \wedge (p_1 \times p_1 \times p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}) = 0$.

Therefore neighbourhoods of p_i are,

$$^{k-1}c_1 + {}^{k-1}c_2 + {}^{k-1}c_3 + \cdots + {}^{k-1}c_{k-1} + {}^{k-2}c_{k-2} + {}^{k-2}c_{k-3} + \cdots + {}^{k-2}c_2 + {}^{k-2}c_1 + 1$$

$$i.e.\ \sum_{r=1}^{k-1} {}^{k-1}c_r + \sum_{r=1}^{k-2} {}^{k-2}c_r + 1.$$

Therefore,

$$|N(p_i)| = \sum_{r=1}^{k-1} {}^{k-1}c_r + \sum_{r=1}^{k-2} {}^{k-2}c_r + 1, \quad i = 2,3,4,\cdots k.$$

We now prove some results for those vertices in $\Gamma(L_n), \Gamma(L_n^{1^2})$ which are dual atoms.

Theorem 4: If L_n be a lattice of all divisors of a positive integer n with respect to divisibility of the form $n = p_1 \times p_2 \times p_3 \times \cdots \times p_k$ (all p_k are distinct) then in the zero-divisor graph of lattice L_n is,

$|N(q)| = 1$, $q = p_{j_1} \times p_{j_2} \times p_{j_3} \times \cdots \times p_{j_r}$, $r = 1,2,3,\cdots k-1$, $j = 1,2,3,\cdots k$.

Proof: Let L_n be a lattice of all divisors of positive integer n with respect to divisibility where $p_1, p_2, p_3, \cdots p_k$ all are prime factors of n and $\Gamma(L_n)$ is the zero-divisor graph of lattice L_n. For $a, b \in \Gamma(L_n)$ if $a \nmid b$ or $b \nmid a$ such that $a \wedge b = 0$ then $a - b$ is an edge in graph. Let $q = p_{j_1} \times p_{j_2} \times p_{j_3} \times \cdots \times p_{j_r} \in L_n$, $r = 1,2,3,\cdots k-1$, $j = 1,2,3,\cdots k$ is a dual atom then there exists one p_i such that

$p_i \nmid q$, $p_i \wedge q = 0$, $i \neq j$, $i,j = 1,2,3,\cdots k$, i.e. $p_i - q$ is an edge in graph.

Therefore neighbourhood of q is one atom p_i, $i = 1,2,3,\cdots k$.

Hence,

$$|N(q)| = 1, \quad q = p_{j_1} \times p_{j_2} \times p_{j_3} \times \cdots \times p_{j_r}, \quad r = 1,2,3,\cdots k-1.$$

Theorem 5: If $L_n^{1^2}$ be a lattice of all divisors of a positive integer n with respect to divisibility of the form $n = p_1^2 \times p_2 \times p_3 \times \cdots \times p_k$ (all p_k are distinct) then in the zero-divisor graph of lattice $L_n^{1^2}$ is,

$$|N(q)| = 0 , q = p_1 \times p_2 \times p_3 \times \cdots \times p_k .$$

Proof: Let $L_n^{1^2}$ be a lattice of all divisors of a positive integer n with respect to divisibility where $p_1 , p_2 , p_3 , \cdots p_k$ all are prime factors of n and $\Gamma(L_n^{1^2})$ is the zero-divisor graph of lattice $L_n^{1^2}$. For $a, b \in \Gamma(L_n^{1^2})$ if $a \nmid b$ or $b \nmid a$ such that $a \wedge b = 0$ then $a - b$ is an edge in graph.

Let $q = p_1 \times p_2 \times p_3 \times \cdots \times p_k \in L_n^{1^2}$ is dual atom then there is no element p_i such that $q \wedge p_i = 0$, since every $p_i \mid q$, $i = 1,2,3,\cdots k$.

Hence,

$|N(q)| = 0$, $q = p_1 \times p_2 \times p_3 \times \cdots \times p_k$.

Theorem 6: If $L_n^{1^2}$ be a lattice of all divisors of a positive integer n with respect to divisibility of the form $n = p_1^2 \times p_2 \times p_3 \times \cdots \times p_k$ (all p_k are distinct) then in the zero-divisor graph of a lattice $L_n^{1^2}$ is,

$|N(q)| = 1$, $q = p_1^2 \times p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}$, $r = 1,2,\cdots k-2$, $j = 2,3,\cdots k$.

Proof: Let $L_n^{1^2}$ be a lattice of all divisors of positive integer n with respect to divisibility where $p_1, p_2, p_3, \cdots p_k$ all are prime factors of n and $\Gamma(L_n^{1^2})$ is the zero-divisor graph of lattice $L_n^{1^2}$. For $a, b \in \Gamma(L_n^{1^2})$ if $a \nmid b$ or $b \nmid a$ such that $a \wedge b = 0$ then $a - b$ is an edge in graph. Let $q = p_1^2 \times p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r} \in L$, $r = 1,2,\cdots k-2$, $j = 2,3,\cdots k$ is dual atom then there exists one p_i, $i = 2,3,\cdots k$ such that,

$$p_i \nmid q , \ p_i \wedge q = 0 , \ i \neq j, \ i = 2,3,\cdots k,$$

i.e. $p_i - q$ is an edge in the graph .

Hence,

$$|N(q)| = 1, \quad q = p_1^2 \times p_{j_1} \times p_{j_2} \times \cdots \times p_{j_r}, \quad r = 1,2,\cdots k-2 , \quad j = 2,3,\cdots k.$$

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