CHARACTERIZATION OF FUZZY NON-BONDAGE NUMBER OF FUZZY GRAPHS

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Abstract: In this paper, we define the non-bondage number bn(G) for any fuzzy graph and study fuzzy cut nodes. A characterization is obtained for fuzzy graphs G such that G* is a cycle. A sufficient condition for a node to be not a fuzzy cut node is obtained which becomes also necessary in the case of fuzzy trees and the exact value of bn(G) for any graph G is found. Moreover we also obtained relationships between bn(G) and b[9].

Keywords: Minimum dominating set γ(G),maximum non-bondage number bn(G) , minimum bondage number b(G), total number of arcs of G is q, total number of arcs of Δ is Δn, total number of arcs of G is p

Introduction: Fuzzy graph theory was introduced by A. Rosenfeld [7] in 1975. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of neural networks, expert systems, information theory, cluster analysis, medical diagnosis, control theory, etc. Sunil Mathew, Sunitha M.S [9] has obtained the fuzzy graph-theoretic concepts like f- bonds, paths, cycles, trees and connectedness and established some of their properties. V.R. Kulli and B. Janakiram[2] have established the non-bondage number of a graph. First we give the definitions of basic concepts of fuzzy graphs and define the non-bondage and it is properties. All graphs consider here are finite, undirected, with no loop or multi arc and p nodes and q (fuzzy) arcs. Any undefined term in this paper may be found in Harary[1]. Among the various applications of the theory of domination that have been considered, the one that is perhaps most often discussed concerns a communication network. Such a network consists of existing communication links between a fixed set of sites. The problem is to select a smallest set of sites at which to place transmitters so that every site in the network that does not have a transmitter is joined by a direct communication link to one that does have a transmitter. This problem reduces to that of finding a minimum dominating set in the graph corresponding to the network. This graph has a node representing each site and an arc between two nodes iff the corresponding sites have a direct communications link joining them. To minimize the direct communication links in the network, we introduce the following section.

2. Preliminaries: A fuzzy subset of a non-empty set V is a mapping σ: V→ [0, 1]. A fuzzy relation on V is a fuzzy subset of E (VxV). A fuzzy graph G = (σ,μ) is a pair of function σ : V→[0,1] and μ : VxV → [0,1], where μ(u, v) ≤ σ(u)Λ σ(v) for all u, v є V. The underlying crisp graph of G=(σ,μ) is denoted by G*=(V,E), where V={u є V:σ(u)>0} and E={(u,v)єVxV:μ(u, v)>0 }. The order p =$\sum_{v\in D}$ σ(v). The graph G = (σ, μ) is denoted by G, if unless otherwise mentioned.

Let be a fuzzy graph on. The degree of a vertex u is$d_G(u) = \sum_{u\neq v} \mu(uv)$. The minimum degree of G is δ(G)=$\Lambda\{d_G(u), \forall v \in V\}$ and the maximum degree of G is $\Delta(G) = \vee \{d_G(u), \forall v \in V\}$

The strength of connectedness between two nodes u and v in a fuzzy graph G is define as the maximum of the strength of all paths between u and v and is denoted by CONNG (u, v). A u-v path P is called a strongest path if its strength equals CONNG (u, v). A path P of length n is a sequence of distinct nodes u0 u1, u2…,un such that (ui-1, ui) > 0 and degree of membership of a weakest arc is defined as its strength. If u0 = un and n ≥ 3, then P is called a cycle and it is a fuzzy cycle if there is more than one weak arc. Let u be a node in fuzzy graphs G then N(u) = {v: (u, v) is strong arc} is called neighborhood of u and N[u] = N(u) U {u} is called closed neighborhood of u. Neighborhood degree of the node is defined by the sum of the weights of the strong neighbor node of u is denoted by ds(u)=$\sum_{v\in N(u)}$ σ(v).

3. Fuzzy dominating set:

Definition 3.1: Let G be a fuzzy graph and u be a node in G then there exist a node v such that (u, v) is a strong arc then u dominates v .

Definition 3.2 : Let G be a fuzzy graph. A subset D of V is said to be a fuzzy dominating set if for every node v є V\D, there exists u є D such that u dominates v.

Definition 3.3:The domination number of G is the minimum cardinality taken over all dominating sets in G and is denoted by γ(G) ,where γ(G) =$\sum_{v\in D}$ σ(v) . *A* dominating set with cardinality γ(G) is called γ - set of G.

4. Fuzzy non bondage number:

Definition 4.1: The fuzzy graph H = (τ, ρ) is called a fuzzy sub graph of G if $\tau(x) \leq$ σ(x)for all x ϵ V and ρ(x,y) ≤ μ (x, y) for all(x, y) ϵ V.

Definition 4.2: A fuzzy sub graph H =(τ, ρ) is said to be a spanning fuzzy sub graph of G, if
$\tau(x) =$ σ(x) for all x.

Definition 4.3: A fuzzy G is said to be connected if there exists a strongest path between any two nodes of G.

Definition4.4: The bondage number b(G) of a fuzzy graph G(V,E,σ, μ)) is minimum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that

$$CONN_{G-(x,y)}(u, v) < CONN_G(u, v)$$

for all u ∈ V- γ(G) and a v ∈γ(G).

Definition 4.5: The non-bondage number bn(G) of a fuzzy graph G(V,E,σ, μ)) is maximum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that $CONN_{G-(x,y)}(u, v) = CONN_G(u, v)$ for all u ∈ V- γ(G) and a v ∈γ(G).

Theorem 4.6: For any fuzzy graph, (x, y) is a non – bondage iff (x, y) is a weakest arc of any cycle. ollowing statemaent are

Proof: Let (x, y) be non bondage arc of G and there exits strongest path between x and y and it is not involving (x, y) if it is involve (x, y) it is form a cycle exits unique strong path. So all its arcs are strictly stronger than μ(x, y) clearly (x, y) is only weakest arc of any cycle.

Conversely, if (x, y) is a weakest arc of a cycle , then any path involving arc (x, y) can be converted into a path not involving (x, y) but at least as strong by using the rest of cycle as a path from x to y thus (x, y) is non-bondage.

Remark 4.7:

1. Let G: (σ, μ)be a fuzzy graph such that G*:(σ^*, μ^*) is a cycle and lett = min{μ(u, v):μ(u, v)> 0}.Then all arcs (u, v)such that μ(u, v)>t are fuzzy bondage of G.
2. μ(u, v)= t is only non-bondage of G

Theorem 4.8: Let G:(σ, μ) be fuzzy graph such that G*:(σ^*, μ^*) is a cycle. Then a node is a not fuzzy cut node of G iff it is incident with a non-bondage arc.

Proof: Let w be a not fuzzy cut node of G. Then there exist u and v other than w such that w is on u-v path. Now G*:(σ^*, μ^*) being a cycle, there exits only one strongest u-v path containing w and by (Remark4.7) all its arcs are fuzzy bondage it is contradiction to assumption. So w must have incident non- bondage arc.

Conversely, let w be incident with non-bondage, so no strongest path of any pair of (u,v) of G does contain w, so strength does not reduce remove from G. Clearly w is not fuzzy cut node, (by definition).

Theorem4.9: If (u, v) is non-bondage then μ(u, v) < CONNG-(x, y)(u, v).

Proof: If (u, v) is non-bondage, we must have CONNG-(x,y)(x, y) = CONNG(x, y) ≥μ(u, v)

Theorem 4.10: If (u, v) is not fuzzy non-bondage then μ(u, v) = CONNG(u, v)

Proof: Suppose that (u,v) us not a fuzzy non-bondage and that μ(u, v) < CONNG(u, v). Then there exists a strongest u-v path with strength greater then μ(u, v) and all arcs of this stronger path have strength greater than (x, y) .Now this path together with the arc (u,v) forms a cycle in which (u, v) is the non-bondage, contradicting the fact that (u, v) is not a fuzzy non-bondage

Remark 4.11: Above result converse are not true generally (distinct edge labelling)

Definition 4.12: A fuzzy graph G=(σ, μ) is called fuzzy forest if it has a fuzzy spanning subgraph F =(σ, v) which is a forest, where all (x, y) not in F, μ(x,y) <v^{∞}(x,y)

In other words, if (x, y) in G but not in F there is a path in F between x and y whose strength is greater than μ(x, y). A connected fuzzy forest is called a fuzzy tree

Note: A fuzzy graph G = (σ, μ) is called fuzzy forest if it has a fuzzy spanning subgraph F =(σ, v) which is a forest ,where all (x, y) not in F,
μ(x,y) <CONNF(x,y)
or
μ(x,y) <CONNG-(x,y)(x,y)

Theorem 4.13: If G is a fuzzy forest, the arcs of F are not fuzzy non -bondage of G

Proof: An arc (x, y) not in F is fuzzy non- bondage since μ(x, y) <v^{∞}(x, y) ≤ CONNG-(x, y)(x, y). Conversely, let (x, y) be an arc in F. If it were fuzzy non – bondage, we would have a path from x to y, such that CONNG-(x, y)(x, y) ≥ μ(x, y). This path must involve arcs not in F, since F is a fuzzy forest and has no cycles. However, by definition, any such arc (u1, v1) can be replaced by a ρ1 path in F of CONNG-(u, v) (u, v) > μ(u, v). Now ρ1 cannot involve (x, y) since all its arcs are strictly stronger than μ(u, v)≥ μ(x, y) .Thus by replacing each (u1,v1) by ρ1, we can construct a path in F from x to y that does not involve (x, y) gives us a cycle in F a contradiction

Remark 4.14:

If above G is connected then G is tree

Theorem 4.15: If G:(σ, μ) is a fuzzy tree iff the following are equivalent.

1. (u, v) is a not non bondage.
2. μ(u, v) = CONNG(u, v)

Proof: Let G:(σ, μ)be a fuzzy tree and let (u, v) be not a fuzzy non-bondage .Then μ(u, v) = CONNG(u, v)(Theorem4.10). Now, let (u, v) be an arc in G such that μ (u, v) = CONNG(u,v).If G* is a tree. Then clearly (u, v) is not a fuzzy non-bondage that (u, v) is in F and (u, v) is not a fuzzy non-bondage (Theorem 4.13).

Conversely, assume that 1↔2 Construct a maximum spanning tree T=(σ, v) for G. If (u, v) is in T thenμ (u,

v) = CONNG (u,v)and hence (u, v) is not a fuzzy non-bondage Now these are the only not fuzzy non-bondage of G for if possible let(u', v')be not a fuzzy non-bondage of G which is not in T. Consider a cycle C consisting of $(u', v')$and the unique $u' - v'$path in T. Now arcs of this$u' - v'$path being not fuzzy non bondage they are not weakest arcs of C and hence can be fuzzy non bondage (Theorem 4.6).

Moreover, for all arcs(u', v') not in T. we have $\mu(u', v') <$CONNT(u', v') for if possible let $\mu(u', v') \geq$CONNT(u', v')But $\mu(u', v') <$ CONNG(u', v') since (u', v') a fuzzy non bondage. So, CONNT$(u', v')<$CONNG(u', v')which gives a contradiction. Since CONNT$(u', v')$is the strength of the unique $u' - v'$ path in T and hence CONNG(u', v') =CONNT(u', v').Thus T is the required spanning sub graph F which is three and hence G is a fuzzy tree.

Theorem4.16: A complete fuzzy graph with n notes has n-1 non bondage

Proof: Need to prove that a complete fuzzy graph with n notes has atmost1 bondage. Let G: (σ, μ) be a complete fuzzy graph with | V | = 3. Then G can have atmost 1fuzzy bondage. Now, let | V | ≥ 4 and let u1, u2, u3 and u4 be any four nodes of G. without loss of generality, let u1 be such that σ(u1) in least among σ(ui)'s i =1, 2, 3, 4. Then (u1, u2) (u1, u3) and (u1, u4) are fuzzy non-bondage , they bring the weakest arcs of some cycle in the fuzzy sub graph induced by u1, u2, u3, u4 Now the arcs (u2, u3), (u2, u4) and (u3, u4) are adjacent to each other and it follows that atmost one of them can be a fuzzy bondage . We also have

Theorem4.17:A complete fuzzy graph has no fuzzy cut nodes

Proof: From theorem 4.8 and 4.16 use to get result

5. Exact values of bn (G) for some standard graphs

Proposition 5.1.: If Ppis a path with p ≥ 4 nodes, then bn(Pp) = $\left\lceil \frac{p}{3} \right\rceil$ - 1.

Proposition5.2: If Cp is a cycle with p≥ 3 nodes, then bn(Cp) = $\left\lceil \frac{p}{3} \right\rceil$.

Proposition5.3: If Kp is a complete graph p≥ 3 nodes, then bn(Kp) = $\frac{(p-1)\,(p-2)}{2}$

Proposition5.4: f Km,n is a complete bipartite graph ,then bn(Km,n) = mn-m-n+2

Proposition5.5 If Wp is a wheel with p≥4 nodes, then bn(Wp) = p-1.

Proposition5.6 : For any tree T, bn(T) = γ(T) -1.

Conclusion: Above non bondage value (≠ 0) is not true for all graphs because K1nor star graph and P3nonbondage value is 0 and also bondage number is equal to number of edges of such above graphs

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