A STUDY ON SINGLE SERVER BULK QUEUE WITH TWO CHOICES OF SERVICE AND COMPULSORY VACATION

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Abstract: This paper deals with customers arriving in bulk or group in a single server queueing system in Poisson distribution which provides two types of general in bulk of fixed size $M(\geq 1)$ or a $\min(n, M)$ in first come first served basis arbitrarily. Once the customer is serviced he leaves the system. In case of the server, the server takes compulsory arbitrary vacation after completion of the service. If the required bulk of customers is not available on the return of the server, the server again goes for vacation or remains in the system till the bulk is reached. We obtain the time dependent probability generating functions and from it the corresponding steady state results are obtained. Also, the average queue size and the system size are briefed.

Keywords: bulk arrival, bulk service, probability generating function, vacation.

1. Introduction: Among the numerous researchers, Chaudhry and Templeton [3], Gross and Harris [7] and Medhi [11] have worked on bulk queues. Vacation queue has been surveyed extensively by Doshi [4]. Shanthikumar [12] and Fuhrmann and Cooper [5] have studied Stochastic Decompositions in the M/G/1 Queue with Generalized Vacations. Thangaraj [14] and Kalyanaraman [8] have analyzed two stage heterogeneous service compulsory server vacations. Also, Madan [9] has detailed single arrival, bulk service with compulsory server vacations. Madan and Walid Abu-Dayyeh [10] have discussed the steady state Analysis of a Single Server Bulk Queue with General Vacation Times. Sikdar and Gupta[13] have talked about batch arrival and batch service with vacations.

The queueing system with bulk arrival and bulk service with compulsory server vacation is explained in this paper. The arrival is under Poisson distribution and the services and vacation are arbitrary. Each batch of customers is provided with two types of services to choose from. The bulk of customers is served under first come first served basis. All the arriving batches are allowed to join the system.

Many popular researchers have discussed bulk queues in the field of queueing theory. Many researchers have shown interest in its applications. Bulk queues have been frequently analyzed. In this concept, the arrivals or departures or both happen in batches of fixed or variable size. In many real time situations, customers served in batches. So, customer served singly does not happen. The concept of this paper has many real-time applications like telecommunications, traffic signal systems, postal service, banking and so on. It is used in areas where customers have choice of choosing our service.

This paper is organized as follows: The mathematical model is briefed in section 2. Definitions and Notations are explained in section 3. Equations governing the system are explained in section 4. The time dependent solutions have been derived in section 5 and corresponding steady state results have been calculated clearly in section 6. The average queue size and the system size are briefed in section 7.

2. The Mathematical Model: We assume the following to describe the queueing model of our study:

(a) Customers (units) arrive at the system in batches of variable size in a compound Poisson process.

(b) Let $\lambda\pi_i dt\ (i = 1,2,3, \dots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq \pi_i \leq 1$, $\sum_{i=1}^{\infty} \pi_i = 1$, $\lambda > 0$ is the mean arrival rate of batches.

(c) We consider the case when there is single server providing parallel service of two types on a first come first served basis (FCFS). At the start of the service, each batch of customers has the choice of choosing either first service with probability θ_1 or the second service with probability θ_2 and $\theta_1 + \theta_2 = 1$.

(d) The service of customers (units) is rendered in batches of fixed size $M(\geq 1)$ or $\min(n, M)$, where n is the number of customers in the queue.

(e) We assume that the random variable of service time $S_j (j = 1,2)$ of the j^{th} kind of service follows a general probability law with distribution function $G_j(s_j)$, $g_j(s_j)$ is the probability density function and $E(S_j^k)$ is the k^{th} moment $(k = 1,2, \dots)$ of service time $j = 1,2$.

(f) Let $\mu_j(x)$ be the conditional probability of type j service completion during the period $(x, x + dx]$, given that elapsed service time is x, so that

$$\mu_j(x) = \frac{g_j(x)}{1-G_j(x)}, j = 1,2 \quad (1)$$

and therefore

$$g_j(s_j) = \mu_j(s_j) e^{-\int_0^{s_j} \mu_j(x)dx}, j = 1,2$$
(2)

(g) After completion of continuous service to the batches of fixed size $M(\geq 1)$, the server will go for compulsory vacation.

(h) We further assume that the random variable of vacation time Y follows a general probability law with distribution function $V(y)$, $v(y)$ is the probability density function and $E(Y^k)$ is the k^{th} moment $(k = 1,2, \dots)$ of vacation time.

(i) Let $\alpha(x)$ be the conditional probability of completion of a vacation period during the interval $(x, x + dx]$, given that the elapsed vacation time is x, so that

$$\alpha(x) = \frac{v(x)}{1-V(x)} \tag{3}$$

and therefore

$$v(y) = \alpha(y)e^{-\int_0^y \alpha(x)dx} \tag{4}$$

(j) On returning from vacation the server instantly starts the service if there is a batch of size M or he remains idle in the system.

(k) All arriving batches are allowed to join the system at all times.

(l) Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

3. Definitions and Notations: We define: $P_{n,j}(x,t)$: Probability that at time t, the server is active providing and there are n $(n \geq 0)$ customers in the queue, excluding a batch of M customers in type j service, $j = 1,2$ and the elapsed service time of this customer is x. Accordingly, $P_{n,j}(t) = \int_0^\infty P_{n,j}(x,t)dx$ denotes the probability that there are n customers in the queue excluding a batch of M customers in type j service, $j = 1,2$ irrespective of the elapsed service time x.

$V_n(x,t)$: Probability that at time t, there are n $(n \geq 0)$ customers in the queue and the server is on vacation with the elapsed vacation time x. Accordingly, $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that there are n customers in the queue and the server is on vacation irrespective of the value of x.

$Q(t)$: Probability at time t, there are less than M customers in the system and the server is idle but available in the system.

4. Equations Governing the System:

According to the Mathematical model mentioned above, the system has the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_{n,1}(x,t) + \frac{\partial}{\partial t}P_{n,1}(x,t) + \big(\lambda + \mu_1(x)\big)P_{n,1}(x,t) = \lambda \sum_{k=1}^{n} \pi_k P_{n-k,1}(x,t) \tag{5}$$

$$\frac{\partial}{\partial x}P_{0,1}(x,t) + \frac{\partial}{\partial t}P_{0,1}(x,t) + \big(\lambda + \mu_1(x)\big)P_{0,1}(x,t) = 0 \tag{6}$$

$$\frac{\partial}{\partial x}P_{n,2}(x,t) + \frac{\partial}{\partial t}P_{n,2}(x,t) + \big(\lambda + \mu_2(x)\big)P_{n,2}(x,t) = \lambda \sum_{k=1}^{n} \pi_k P_{n-k,2}(x,t) \tag{7}$$

$$\frac{\partial}{\partial x}P_{0,2}(x,t) + \frac{\partial}{\partial t}P_{0,2}(x,t) + \big(\lambda + \mu_2(x)\big)P_{0,2}(x,t) = 0 \tag{8}$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) + \big(\lambda + \alpha(x)\big)V_n(x,t) = \lambda \sum_{k=1}^{n} \pi_k V_{n-k}(x,t) \tag{9}$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) + \big(\lambda + \alpha(x)\big)V_0(x,t) = 0 \tag{10}$$

$$\frac{d}{dt}Q(t) + \lambda Q(t) = \int_0^\infty V_0(x,t)\alpha(x)dx \tag{11}$$

Equations (5) – (11) are to be solved subject to the following boundary conditions:

$$P_{n,1}(0,t) = \theta_1 \int_0^\infty V_{n+M}(x,t)\alpha(x)dx \tag{12}$$

$$P_{0,1}(0,t) = \theta_1 \sum_{b=1}^{M} \int_0^\infty V_b(x,t)\alpha(x)dx + \theta_1 \lambda Q(t) \tag{13}$$

$$P_{n,2}(0,t) = \theta_2 \int_0^\infty V_{n+M}(x,t)\alpha(x)dx \tag{14}$$

$$P_{0,2}(0,t) = \theta_2 \sum_{b=1}^{M} \int_0^\infty V_b(x,t)\alpha(x)dx + \theta_2 \lambda Q(t) \tag{15}$$

$$V_n(0,t) = \int_0^\infty P_{n,1}(x,t)\mu_1(x)dx + \int_0^\infty P_{n,2}(x,t)\mu_2(x)dx \tag{16}$$

We assume that initially the server is available but idle because of less than M customers so that the initial conditions are

$$V_n(0) = 0;\ V_0(0) = 0;\ Q(0) = 1$$
$$P_{n,j}(0) = 0, \text{for } n = 0,1,2, \dots \text{ and } j = 1,2. \tag{17}$$

5. Probability Generating Function of The Queue Size: The Transient Solution

We define the following probability generating functions:

$$\left.\begin{aligned}
P_j(x,z,t) &= \sum_{n=0}^{\infty} P_{n,j}(x,t)z^n, \ j = 1,2\\
P_j(z,t) &= \sum_{n=0}^{\infty} P_{n,j}(t)z^n, \ j = 1,2\\
V(x,z,t) &= \sum_{n=0}^{\infty} V_n(x,t)z^n\\
V(z,t) &= \sum_{n=0}^{\infty} V_n(t)z^n\\
\pi(z) &= \sum_{n=1}^{\infty} \pi_n z^n
\end{aligned}\right\} \tag{18}$$

Define the Laplace-Stieltjes Transform of a function $f(t)$ as follows:

$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \tag{19}$$

Taking Laplace Transform of equations (5) – (16) and using (17), we get,

$$\frac{\partial}{\partial x}\bar{P}_{n,1}(x,s) + \big(s + \lambda + \mu_1(x)\big)\bar{P}_{n,1}(x,s) = \lambda \sum_{k=1}^{n} \pi_k \bar{P}_{n-k,1}(x,s) \tag{20}$$

$$\frac{\partial}{\partial x}\bar{P}_{0,1}(x,s)+\big(s+\lambda+\mu_1(x)\big)\bar{P}_{0,1}(x,s)=0 \quad (21)$$

$$\frac{\partial}{\partial x}\bar{P}_{n,2}(x,s)+\big(s+\lambda+\mu_2(x)\big)\bar{P}_{n,2}(x,s)=\lambda\sum_{k=1}^{n}\pi_k\bar{P}_{n-k,2}(x,s) \quad (22)$$

$$\frac{\partial}{\partial x}\bar{P}_{0,2}(x,s)+\big(s+\lambda+\mu_2(x)\big)\bar{P}_{0,2}(x,s)=0 \quad (23)$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s)+\big(s+\lambda+\alpha(x)\big)\bar{V}_n(x,s)=\lambda\sum_{k=1}^{n}\pi_k\bar{V}_{n-k}(x,s) \quad (24)$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s)+\big(s+\lambda+\alpha(x)\big)\bar{V}_0(x,s)=0 \quad (25)$$

$$(s+\lambda)\bar{Q}(s)=1+\int_0^\infty\bar{V}_0(x,s)\alpha(x)dx \quad (26)$$

$$\bar{P}_{n,1}(0,s)=\theta_1\int_0^\infty\bar{V}_{n+M}(x,s)\alpha(x)dx \quad (27)$$

$$\bar{P}_{0,1}(0,s)=\theta_1\sum_{b=1}^{M}\int_0^\infty\bar{V}_b(x,s)\alpha(x)dx+\theta_1\lambda\bar{Q}(s) \quad (28)$$

$$\bar{P}_{n,2}(0,s)=\theta_2\int_0^\infty\bar{V}_{n+M}(x,s)\alpha(x)dx \quad (29)$$

$$\bar{P}_{0,2}(0,s)=\theta_2\sum_{b=1}^{M}\int_0^\infty\bar{V}_b(x,s)\alpha(x)dx+\theta_2\lambda\bar{Q}(s) \quad (30)$$

$$\bar{V}_n(0,s)=\int_0^\infty\bar{P}_{n,1}(x,s)\mu_1(x)dx+\int_0^\infty\bar{P}_{n,2}(x,s)\mu_2(x)dx \quad (31)$$

Multiplying the equation (20) by z^n and summing over n from 1 to ∞, adding equation (21) and using the generating functions defined in (18), we obtain,

$$\frac{\partial}{\partial x}\bar{P}_1(x,z,s)+\{s+\lambda\big(1-\pi(z)\big)+\mu_1(x)\}\bar{P}_1(x,z,s)=0 \quad (32)$$

Performing similar operations on equations (22) – (25), we obtain,

$$\frac{\partial}{\partial x}\bar{P}_2(x,z,s)+\{s+\lambda\big(1-\pi(z)\big)+\mu_2(x)\}\bar{P}_2(x,z,s)=0 \quad (33)$$

$$\frac{\partial}{\partial x}\bar{V}(x,z,s)+\{s+\lambda\big(1-\pi(z)\big)+\alpha(x)\}\bar{V}(x,z,s)=0 \quad (34)$$

Multiplying the equation (27) by z^{n+M} and summing over n from 1 to ∞ and adding, multiplying the equation (28) by z^M, and using the generating functions defined in (18), we obtain,

$$z^M\bar{P}_1(0,z,s)=\theta_1\int_0^\infty\bar{V}(x,z,s)\alpha(x)dx+\theta_1\lambda\bar{Q}(s)z^M-\theta_1\int_0^\infty\bar{V}_0(x,s)\alpha(x)dx-\theta_1\sum_{b=1}^{M-1}\int_0^\infty\bar{V}_b(x,s)\alpha(x)dx\,z^b+\theta_1\sum_{b=1}^{M-1}\int_0^\infty\bar{V}_b(x,s)\alpha(x)dx\,z^M \quad (35)$$

Using (26), we obtain,

$$\bar{P}_1(0,z,s)=\theta_1z^{-M}U^*+\theta_1z^{-M}\int_0^\infty\bar{V}(x,z,s)\alpha(x)dx+\theta_1z^{-M}+\theta_1\big(\lambda-(s+\lambda)z^{-M}\big)\bar{Q}(s) \quad (36)$$

Where $U^*=\sum_{b=1}^{M-1}(z^M-z^b)\int_0^\infty V_b(x)\alpha(x)dx$

Performing similar operations on equations (29) and (30), we obtain,

$$\bar{P}_2(0,z,s)=\theta_2z^{-M}U^*+\theta_2z^{-M}\int_0^\infty\bar{V}(x,z,s)\alpha(x)dx+\theta_2z^{-M}+\theta_2\big(\lambda-(s+\lambda)z^{-M}\big)\bar{Q}(s) \quad (37)$$

Multiplying the equation (31) by z^n and summing over n from 0 to ∞ and using the generating functions defined in (18), we obtain,

$$\bar{V}(0,z,s)=\int_0^\infty\bar{P}_1(x,z,s)\mu_1(x)dx+\int_0^\infty\bar{P}_2(x,z,s)\mu_2(x)dx \quad (38)$$

We now integrate equations (32) – (34) between the limits 0 and x and obtain,

$$\bar{P}_1(x,z,s)=\bar{P}_1(0,z,s)e^{-Rx-\int_0^x\mu_1(x)dx} \quad (39)$$

$$\bar{P}_2(x,z,s)=\bar{P}_2(0,z,s)e^{-Rx-\int_0^x\mu_2(x)dx} \quad (40)$$

$$\bar{V}(x,z,s)=\bar{V}(0,z,s)e^{-Rx-\int_0^x\alpha(x)dx} \quad (41)$$

Where $R=s+\lambda\big(1-\pi(z)\big)$

Integrating equations (39) – (41) by parts, with respect to x, we get,

$$\bar{P}_1(z,s)=\bar{P}_1(0,z,s)\left[\frac{1-\bar{G}_1(R)}{R}\right] \quad (42)$$

$$\bar{P}_2(z,s)=\bar{P}_2(0,z,s)\left[\frac{1-\bar{G}_2(R)}{R}\right] \quad (43)$$

$$\bar{V}(z,s)=\bar{V}(0,z,s)\left[\frac{1-\bar{V}(R)}{R}\right] \quad (44)$$

Where $\bar{G}_j(R)=\int_0^\infty e^{-R\,x}dG_j(x)$, is the Laplace Transform of j^{th} type of service, $j=1,2$.

$\bar{V}(R)=\int_0^\infty e^{-R\,x}dV(x)$ is the Laplace Transform of vacation time.

Multiplying the equations (39), (40) and (41) by $\mu_1(x)$, $\mu_2(x)$and $\alpha(x)$ integrating by parts, with respect to x, we get,

$$\int_0^\infty\bar{P}_1(x,z,s)\mu_1(x)dx=\bar{P}_1(0,z,s)\bar{G}_1(R) \quad (45)$$

$$\int_0^\infty\bar{P}_2(x,z,s)\mu_2(x)dx=\bar{P}_2(0,z,s)\bar{G}_2(R) \quad (46)$$

$$\int_0^\infty\bar{V}(x,z,s)\alpha(x)dx=\bar{V}(0,z,s)\bar{V}(R) \quad (47)$$

Substituting (47) in (36) we get,

$$\bar{P}_1(0,z,s)=\theta_1\bar{V}(0,z,s)\bar{V}(R)z^{-M}+\theta_1z^{-M}+\theta_1z^{-M}U^*+\theta_1\big(\lambda-(s+\lambda)z^{-M}\big)\bar{Q}(s) \quad (48)$$

Substituting (45), (46) in (38) we get,

$$\bar{V}(0,z,s)=\bar{P}_1(0,z,s)\bar{G}_1(R)+\bar{P}_2(0,z,s)\bar{G}_2(R) \quad (49)$$

Substituting (49) in (48) we get,

$\bar{P}_1(0,z,s)=\theta_1z^{-M}U^*+\theta_1z^{-M}$

$$+\theta_1\begin{pmatrix}\bar{P}_1(0,z,s)\bar{G}_1(R)\\+\bar{P}_2(0,z,s)\bar{G}_2(R)\end{pmatrix}\bar{V}(R)z^{-M}$$
$$+\theta_1(\lambda-(s+\lambda)z^{-M})\bar{Q}(s) \quad (50)$$

Performing similar operations on (37), we obtain,

$$\bar{P}_2(0,z,s)=\theta_2 z^{-M}U^*+\theta_2 z^{-M}$$
$$+\theta_2\begin{pmatrix}\bar{P}_1(0,z,s)\bar{G}_1(R)\\+\bar{P}_2(0,z,s)\bar{G}_2(R)\end{pmatrix}\bar{V}(R)z^{-M}$$
$$+\theta_2(\lambda-(s+\lambda)z^{-M})\bar{Q}(s) \quad (51)$$

Solving (50) and (51), we get,

$$\bar{P}_1(0,z,s)=\frac{\theta_1\{U^*+T\bar{Q}(s)+1\}}{z^M-\bar{V}(R)\left(\theta_1\bar{G}_1(R)+\theta_2\bar{G}_2(R)\right)} \quad (52)$$

$$\bar{P}_2(0,z,s)=\frac{\theta_2\{U^*+T\bar{Q}(s)+1\}}{z^M-\bar{V}(R)\left(\theta_1\bar{G}_1(R)+\theta_2\bar{G}_2(R)\right)} \quad (53)$$

Where $T=\lambda z^M-(s+\lambda)$

Substituting (52) and (53) in (42) and (43) respectively, we get,

$$\bar{P}_1(z,s)=\left[\frac{\theta_1\{U^*+T\bar{Q}(s)+1\}}{z^M-\bar{V}(R)\left(\theta_1\bar{G}_1(R)+\theta_2\bar{G}_2(R)\right)}\right]\left[\frac{1-\bar{G}_1(R)}{R}\right] \quad (54)$$

$$\bar{P}_2(z,s)=\left[\frac{\theta_2\{U^*+T\bar{Q}(s)+1\}}{z^M-\bar{V}(R)\left(\theta_1\bar{G}_1(R)+\theta_2\bar{G}_2(R)\right)}\right]\left[\frac{1-\bar{G}_2(R)}{R}\right] \quad (55)$$

Substituting (49) in (44) we get,

$$\bar{V}(z,s)=\begin{bmatrix}\bar{P}_1(0,z,s)\bar{G}_1(R)\\+\bar{P}_2(0,z,s)\bar{G}_2(R)\end{bmatrix}\left[\frac{1-\bar{V}(R)}{R}\right] \quad (56)$$

Substituting (52), (53) in (56), we get,

$$\bar{V}(z,s)=\left[\frac{\begin{Bmatrix}\theta_1\bar{G}_1(R)\\+\theta_2\bar{G}_2(R)\end{Bmatrix}z^{-M}U^*+T\bar{Q}(s)+1}{z^M-\bar{V}(R)\left(\theta_1\bar{G}_1(R)+\theta_2\bar{G}_2(R)\right)}\right]\left[\frac{1-\bar{V}(R)}{R}\right] \quad (57)$$

We note that there are M unknowns, $\bar{Q}(s)$and $\bar{V}_b(x,s), b=1,2,\dots,M-1$ appearing in equations (54), (55) and (57).

By Rouche's theorem of complex variables, the denominator of right hand side of equations (54), (55) and (57) has M zeroes inside the contour $|z|=1$. Since $\bar{P}_1(z,s),\bar{P}_2(z,s)$ and $\bar{V}(z,s)$ are analytic inside the unit circle $|z|=1$, the numerator in the right hand side of equations (54), (55) and (57) must vanish at these points, which gives rise to a set of M linear equations which are sufficient to determine Munknowns.

6. The Steady State Results: To define the steady state probabilities and corresponding generating functions, we drop the argument t, and for that matter the argument s wherever it appears in the time-dependent analysis up to this point. Then the corresponding steady state results can be obtained by using the well known Tauberian Property

$$\lim_{s\to 0} s\bar{f}(s)=\lim_{t\to\infty} f(t) \quad (58)$$

if the limit on the right exists.

Now (55), (56) and (57) we have,

$$P_1(z)=\frac{\{\theta_1 U^*+\theta_1 TQ\}\left\{\frac{1-\bar{G}_1(f(z))}{f_1(z)}\right\}}{z^M-\bar{V}(f(z))\left(\theta_1\bar{G}_1(f(z))+\theta_2\bar{G}_2(f(z))\right)} \quad (59)$$

$$P_2(z)=\frac{\{\theta_2 U^*+\theta_2 TQ\}\left\{\frac{1-\bar{G}_2(f(z))}{f(z)}\right\}}{z^M-\bar{V}(f(z))\left(\theta_1\bar{G}_1(f(z))+\theta_2\bar{G}_2(f(z))\right)} \quad (60)$$

$$V(z)=\frac{\begin{Bmatrix}\theta_1\bar{G}_1(f(z))\\+\theta_2\bar{G}_2(f(z))\end{Bmatrix}\{U^*+TQ\}\left\{\frac{1-\bar{V}(f(z))}{f(z)}\right\}}{z^M-\bar{V}(f(z))\left(\theta_1\bar{G}_1(f(z))+\theta_2\bar{G}_2(f(z))\right)} \quad (61)$$

The M unknowns, Q and $\int_0^\infty V_b(x)\alpha(x)dx, b=1,2,\dots,M-1$can be determined as before.

Where $f(z)=\lambda(1-\pi(z)), T=\lambda(z^M-1)$

Let $A_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system.

i. e., $A_q(z)=P_1(z)+P_2(z)+V(z) \quad (62)$

In order to find Q, we use the normalization condition

$$A_q(1)+Q=1$$

Note that for $z=1$, $A_q(1)$is indeterminate of $\frac{0}{0}$ form.

Therefore, we apply L'Hôpital's Rule on (62), we get,

$$A_q(1)=\frac{[E^*+E(V)][U+\lambda MQ]}{M-\lambda E(I)\{E^*+E(V)\}} \quad (63)$$

We used $\bar{G}_j(0)=1,\ j=1,2,\ \bar{V}(0)=1,\ \pi'(1)=E(I)$, where$I$ denotes the number of customers in an arriving batch and therefore, $E(I)$ is the mean of the batch size of the arriving customers. Similarly $E(S_1),\ E(S_2)$ and $E(V)$are the mean service times of type 1, type 2 services and vacation time, respectively.

Where $E^*=\theta_1 E(S_1)+\theta_2 E(S_2)$ and

$$U=\sum_{b=1}^{M-1}(M-b)\int_0^\infty V_b(x)\alpha(x)dx$$

Therefore, adding Q to equation (62) and equating to 1 and simplifying we get,

$$Q=1-\frac{(E^*+E(V))(M\lambda+U)}{M+(M\lambda-\lambda E(I))(E^*+E(V))} \quad (64)$$

Equation (64) gives the probability that the server is idle.

From equation (64) the utilization factor, ρ of the system is given by

$$\rho=\frac{(E^*+E(V))(M\lambda+U)}{M+(M\lambda-\lambda E(I))(E^*+E(V))} \quad (65)$$

Where $\rho<1$ is the stability condition under which the steady state exists.

7. The Average Queue Size and The System Size: Let $L_q$denote the mean number of customers in the queue under the steady state. Then

$$L_q=\frac{d}{dz}A_q(z)\Big|_{z=1} \quad (66)$$

Since the formula gives $\frac{0}{0}$ form, then we write $A_q(z)$ as

$$A_q(z)=\frac{N(z)}{D(z)}$$

Where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of $A_q(z)$ respectively.

Then using L'Hôpital's Rule twice we obtain,

$$L_q=\lim_{z\to 1}\frac{D'(z)N''(z)-N'(z)D''(z)}{2\left(D'(z)\right)^2} \quad (67)$$

We have,

$$N'(1)=[E^*+E(V)][U+\lambda r_1 MQ] \quad (68)$$
$$D'(1)=M-\lambda E(I)\{r_1E^*+E(V)\} \quad (69)$$
$$N''(1)=\lambda E(I)\begin{bmatrix}E^{**}+E(V^2)\\+2E(V)E^*\end{bmatrix}(U+\lambda MQ)$$

$$+[E^* + E(V)](U^{**} + \lambda M(M-1)Q) \quad (70)$$

$$D''(1) = M(M-1) - \lambda^2\big(E(I)\big)^2 \begin{Bmatrix} 2E(V)E^* + \\ E^{**} + E(V^2) \end{Bmatrix}$$

$$-\lambda E\big(I(I-1)\big)\{E^* + E(V)\} \quad (71)$$

Where $E^{**} = \theta_1 E(S_1^2) + \theta_2 E(S_2^2)$ and

$$U^{**} = \sum_{b=1}^{M-1} \begin{pmatrix} M(M-1) \\ -b(b-1) \end{pmatrix} \int_0^\infty V_b(x)\alpha(x)dx$$

Where $E\big(I(I-1)\big)$ is the second factorial moment of the batch size of the arriving customers.

Similarly, $E(S_1^2), E(S_2^2)$ are the second moments of the service times of type 1, type 2 services, respectively. $E(V^2)$ is the second moment of the vacation time and Q has been obtained in (64). Then if we substitute the equations (68), (69), (70), (71) in to the equation (67), we obtain L_q in a closed form. Further, the average number of customers in the system could be found as $L_s = L_q + \rho$ by using Little's formula.

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