

**INDEPENDENT DOMINATION OF LINE GRAPH- LAPLACIAN APPROACH**

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**Abstract:** The Line graph of  $G$ , written  $L(G)$ , is the simple graph whose vertices are the edges of  $G$ , with  $ef \in E(L(G))$  when  $e$  and  $f$  have a common end vertex in  $G$ . An independent dominating set in a graph is a set that is both dominating and independent. The independent domination number of  $G$  is the minimum size of an independent dominating set. In this paper, For connected graph of  $G, i_L(G) \geq \text{number of vertices in } G - \text{Rank of the Laplacian matrix} + Q(\frac{\text{number of internal vertices in longest path in } G}{3})$  where  $Q$  is the quotient.

**Keywords:** domination number, independent domination number, Laplacian matrix, Line graph, Rank.

**Introduction:** The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it because of its many and varied applications in such fields as linear algebra and optimization, design and analysis of communication networks, and social sciences and military surveillance. Many variants of dominating models are available in the existing literature. For a comprehensive bibliography of papers on the concept of domination, readers are referred to Hedetniemi and Laskar. The present paper is focused on independent domination number of Line graphs in terms of vertices and Rank of Laplacian matrix.

**Preliminaries: 2:** In this section, the basic definitions and known results related to our results are given.

**Definition : 2.1:** A graph  $G$  is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of  $G$ , called edges. The vertex set and the edge set of  $G$  are denoted by  $V(G)$  and  $E(G)$  respectively.

If  $e = \{u, v\}$  is an edge, we write  $e = uv$ ; we say that  $e$  joins the vertices  $u$  and  $v$ ;  $u$  and  $v$  are adjacent vertices;  $u$  and  $v$  are incident with  $e$ .

If two vertices are not joined, then we say that they are not joined, then we say that they are non-adjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other.

**Definition: 2.2:** Let  $G = (V, E)$  be a graph. A set  $S$ , subset of  $V$ , is called a dominating set if every vertex in  $V - S$  is adjacent to some vertex in  $S$ . The domination number  $\gamma(G)$  is the minimum cardinality taken over all minimal dominating sets in  $G$ .

**Definition: 2.3:** A set is independent dominating if it is independent and dominating. Let  $i(G)$  denotes the minimum cardinality of an independent dominating set of  $G$ .

**Definition : 2.4:** The adjacency matrix  $A(G)$  of  $G$  is an  $n \times n$  matrix with its row and columns indexed by  $V(G)$  and with the  $(i, j)$ -entry equal to 1 if vertices  $i, j$  are adjacent (i.e., joined by an edge) and 0 otherwise. Thus  $A(G)$  is a symmetric matrix with its  $i$ -th row (or column) sum equal to  $d_i(G)$ , which by definition is the degree of the vertex  $i, i = 1, 2, \dots, n$ .

Let  $D(G)$  denote the  $n \times n$  diagonal matrix, whose  $i$ -th diagonal entry is  $d_i(G), i = 1, 2, \dots, n$ .

The Laplacian matrix of  $G$ , denoted by  $L(G)$ , is simply the matrix  $D(G) - A(G)$ .

**Definition : 2.5:** A graph  $G$  is said to be connected if any two distinct vertices of  $G$  are joined by a path.

**Definition: 2.6:** A set of vertices in a graph  $G$  is called independent set if no two vertices in the set are adjacent.

**Definition: 2.7:** Let  $u$  and  $v$  be vertices of a graph  $G$ . A  $u$ - $v$  walk of  $G$  is a finite, alternating sequence of vertices and edges beginning with vertex  $u$  and ending with vertex  $v$  such that  $e_i = u_{i-1}u_i, i = 1, 2, \dots, n$ . The number  $n$  is called the length of the walk. The walk is said to be open if  $u$  and  $v$  are distinct vertices; it is closed otherwise. A walk  $u_0, e_1, u_1, e_2, u_2, \dots, u_n$  is determined by the sequence  $u_0, u_1, u_2, \dots, u_n$  of its vertices and hence we specify the walk by  $(u_0, u_1, u_2, \dots, u_n)$ . A walk in which all the vertices are distinct is called a path. A closed walk  $u_0, u_1, u_2, \dots, u_n$  in which  $u_0, u_1, u_2, \dots, u_{n-1}$  are distinct is called a cycle. A path on  $n$  vertices is denoted by  $P_n$  and a cycle on  $n$  vertices is denoted by  $C_n$ .

**Definition: 2.8:** Let  $v \in G$  be a vertex a graph  $G$ . The neighbourhood of  $v$  is the set  $NG(v) = \{u \in G \mid vu \in E\}$ .

The degree of  $v$  is the number of its neighbours:  $dG(v) = |NG(v)|$ .

If  $dG(v) = 0$ , then  $v$  is said to be isolated in  $G$ , and if  $dG(v) = 1$ , then  $v$  is a leaf of the graph. The minimum degree and the maximum degree of  $G$  are defined as  $d(G) = \min\{dG(v) \mid v \in G\}$  and  $D(G) = \max\{dG(v) \mid v \in G\}$ .

**Definition :2.9:** The Line graph of  $G$ , written  $L(G)$ , is the simple graph whose vertices are the edges of  $G$ , with  $ef \in E(L(G))$  when  $e$  and  $f$  have a common end vertex in  $G$ .

**Definition:2.10:** Let  $G$  be a connected graph. The distance between two vertices  $u$  and  $v$  in  $G$ , denoted by  $d(u, v)$  is the length of a shortest path between them in  $G$ .

The eccentricity of a vertex  $v$  is  $\text{ecc}(v) := \max_{x \in V(G)} d(v, x)$ . The diameter of  $G$  is  $\text{diam}(G) := \max_{x \in V(G)} \text{ecc}(x)$ .

The radius of  $G$  is  $\text{rad}(G) := \min_{x \in V(G)} \text{ecc}(x)$ . Distance between a vertex  $v$  and a set  $S \subseteq V(G)$  is  $d(v, S) := \min_{x \in S} d(v, x)$ . The  $k$ -step open neighbourhood of a set  $S \subseteq V(G)$  is  $N_k(S) := \{x \in V(G) | d(x, S) = k\}$ ,  $k \in \{0, 1, 2, \dots\}$ . The degree of a vertex  $v$  is  $\text{degree}(v) := |N_1(\{v\})|$ . The minimum degree of  $G$  is  $\delta(G) := \min_{x \in V(G)} \text{degree}(x)$ . A vertex is called pendant if its degree is 1 and isolated if its degree is 0.

**Definition:2.11:** The Gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle.

The Gear graph  $G_n$  has  $2n+1$  vertices and  $3n$  edges.

**Theorem:2.1:** For any graph,  $i_M(G) \geq \lfloor \frac{p}{2} \rfloor$

**Theorem:2.2:** If  $G$  is connected, then the rank of Laplacian matrix is  $n-1$ .

**Main Results:**

**Theorem : 1**

For Star graph, Cycle graph, Path graph, Tree, Unicyclic graph,

$i_L(G) \geq$  number of vertices in  $G$  - Rank of the Laplacian matrix +

$$Q\left(\frac{\text{number of internal vertices in longest path in } G}{3}\right)$$

**Proof:**

Let  $G = (p, q)$  graph. Let  $\{v_1, v_2, v_3, \dots, v_n\}$  be the vertices of  $G$ .

Let  $\{e_1, e_2, \dots, e_n\}$  be the edges of  $G$ . Let  $\{f_1, f_2, f_3, \dots, f_n\}$  be the corresponding vertices of  $L(G)$ .

If  $p \leq 4$ , the result is obvious.

Assume that  $p > 4$

**Case: 1:** If  $\Delta(G) = p-1$ .

If  $G$  contains an edge  $e \in E(G)$  such that  $\text{deg } e = q-1$ .

Then the vertex having maximum degree dominates all the vertices of  $L(G)$

Then  $i(L(G)) = 1$ .

Hence the result holds.

**Case 2:** If  $G$  contains an edge  $e \in E(G)$  such that  $\Delta(G) < p-1$  and if  $\Delta(G) = 2$

If  $G$  is cycle and  $p \equiv 1, 2 \pmod{3}$

Let  $\{f_2, f_3, f_4, \dots, f_i / d(f_i, f_j) = 3, 1 \leq i < j \leq n, f_i, f_j \text{ are not adjacent}\}$  which is independent dominating set of  $L(G)$  with cardinality  $Q\left(\frac{n}{3}\right)$ .

If  $p \equiv 0 \pmod{3}$

Choose  $I = \{f_2, f_5, f_8, \dots, f_i / d(f_i, f_j) = 3, 1 \leq i < j \leq n, f_i, f_j \text{ are not adjacent}\} \cup v_n$  which is independent dominating set of  $L(G)$  with cardinality  $Q\left(\frac{n}{3}\right) + 1$  vertices.

Consider,  $Q\left(\frac{n}{3}\right) + 1 \geq$

$$Q\left(\frac{\text{number of internal vertices in longest path in } G}{3}\right) + 1 \geq Q\left(\frac{\text{number of internal vertices in longest path in } G}{3}\right) + n - n + 1$$

$$\geq Q\left(\frac{\text{number of internal vertices in longest path in } G}{3}\right) + n + (n - 1)$$

$\geq n + Q\left(\frac{\text{number of internal vertices in longest path in } G}{3}\right)$  - Rank of the Laplacian matrix.

**Case 3:** If  $G$  is a path, and if  $n \equiv 2 \pmod{3}$

Choose  $I = \{f_2, f_5, f_8, \dots, f_n / d(f_i, f_{i+3}) = 3, 2 \leq i \leq n\} \cup \{f_n\}$

Clearly,  $I$  is an independent dominating set with cardinality  $Q\left(\frac{n}{3}\right) + 1$  vertices.

If  $n \equiv 0, 1 \pmod{3}$ ,

Choose  $I = \{f_2, f_5, f_8, \dots, f_n / d(f_i, f_{i+3}) = 3, 2 \leq i \leq n\}$

Clearly,  $I$  is an independent dominating set with cardinality  $Q\left(\frac{n}{3}\right)$  vertices.

**Case: 4:** If  $2 < \Delta(G) < p-1$ , If  $G$  is a tree,

Let  $\{u_1, u_2, u_3, \dots, u_{n-1}\}$  be the cutvertices of  $L(G)$

Let  $S = \{u_i, u_j / u_i \text{ and } u_j \text{ are not adjacent } 1 \leq i < j \leq n-1\}$

Let  $I = \{f_i, f_j / u_i \in S \text{ and } d(u_i, x_i) = 2, u_i, x_i \text{ are not adjacent and } x_i, x_j \text{ are not adjacent } 1 \leq i < j \leq n-1\}$

Clearly,  $H = S \cup I$  is an independent dominating set with cardinality  $|H| = \text{diam } G - 1$

Now, Number of internal vertices of  $G$  in longest path is  $\text{diam } G - 1$

$|H| \geq \text{diam } G \geq 1 +$

$$Q\left(\frac{\text{Number of internal vertices in longest path of } G}{3}\right) \geq 1 + Q\left(\frac{\text{Number of internal vertices in longest path of } G}{3}\right)$$

$$\geq Q\left(\frac{\text{Number of internal vertices in longest path of } G}{3}\right) + n - (n-1).$$

**Case 5:** Let  $G$  be Unicyclic graph.

Let  $C = \{e_1, e_2, e_3, \dots, e_n\}$  be the cycle of  $G$ .

Let  $C_1 = \{e_i, e_j / d(e_i, e_j) \text{ are not adjacent, } 1 \leq i < j \leq n\}$  be the edges of cycle with cardinality at most  $Q\left(\frac{n}{2}\right)$  edges.

Consider The connected spanning tree  $T$  of  $G$  by removing the  $Q\left(\frac{n}{2}\right)$  edges of  $C_1$

Clearly,  $T$  is tree of  $G$  with at most  $p - Q\left(\frac{n}{2}\right)$  vertices

$L(T)$  has independent dominating set  $I$  with  $|I| \geq Q\left(\frac{\text{Number of internal vertices in longest path of } G}{3}\right) +$

$n - (n-1)$ .

Now, Adding the edges,

If edges of  $C_1$  adjacent to some vertex of  $L(T)$ , then there is nothing to prove.

If not,  $i(L(G))$

$$\geq Q\left(\frac{\text{Number of internal vertices in longest path of } G}{3}\right) + n - (n-1) + (n-1) + Q\left(\frac{n}{2}\right)$$

$i_L(G) \geq$  number of vertices in  $G$  - Rank of the Laplacian matrix +

$$Q\left(\frac{\text{number of internal vertices in longest path in } G}{3}\right)$$

**Theorem :2**

For integer  $n$ ,  $G \equiv G_n, n \geq 3$ ,

$$Y_c(G) = n - Q\left(\frac{2n+1}{4}\right) + 1$$

**Proof:** Let  $G_n$  be gear graph. It has  $2n+1$  vertices and it contains cycle of length  $n$  vertices.

Let  $u$  be the center vertex

Consider,  $C: v_0, v_1, v_2, v_3, v_4, \dots, v_n = v_0$  forms a cycle of length  $n$ .

$S = \{v_1, v_3, v_5, v_7, \dots, v_{n-1}\}$  with cardinality  $n - Q\left(\frac{2n+1}{4}\right)$ .

$H = \cup\{u\}$  forms a connected dominating set with cardinality  $n - Q\left(\frac{2n+1}{4}\right) + 1$  vertices.

Hence,  $Y_c(G) = n - Q\left(\frac{2n+1}{4}\right) + 1$ .

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