APPROXIMATION OF CONJUGATE OF FUNCTIONS BELONGING TO WEIGHTED LIPSCHITZ CLASS BY $(E, q)(C,\alpha,\beta)$ MEANS OF CONJUGATE FOURIER SERIES

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Abstract: The study of approximation properties of the periodic functions in *Lp*($p \geq 1$)-spaces, in general and in Lipschitz classes such as *Lip* α , *Lip(* α *,p), Lip(* ξ *(t); p)* and weighted Lipschitz class-*W*($L^p, \xi(t), \gamma$), in particular, through conjugate trigonometric Fourier series, although is an old problem and known as conjugate trigonometric Fourier approximation in the existing literature, has been of a growing interests over the last four decades due to its application in filters and signals [E. Z. Psarakis and G. V. Moustakides, An L_2-based method for the design of 1-D zero phase FIR digital filters, IEEE Trans. Circuits Systems I Fundam: Theory Appl:; 44 (7) (1997) 551-601]. The most common methods used for the determination of the degree of approximation of periodic functions are based on the minimization of the L_p-norm of f(x) – T_n(x), where T_n(x) is a trigonometric polynomial of degree *n* and called approximant of the function f. In this paper, we discuss the degree of approximation of $\tilde{f}$, conjugate of a 2π-periodic function f belonging to the weighted *W*($L^p, \xi(t), \gamma \geq 0$)-class and its sub-classes such as *Lip(* ξ *(t); p), Lip(* α *,p)* and *Lip* α by using *(E, q)*(C, α , β) means of conjugate Fourier series of *f*, The degree of approximation obtained in our theorems of this paper is free from p and sharper than earlier results.

Keywords: Conjugate Fourier series, Degree of approximation, Signals, *W*($L^p, \xi(t), \gamma \geq 0$)-class of functions.

Introduction: For a 2π-periodic function $f \in L^P := L^P[0,2\pi], P \geq 1$, integrable in the sense of Lebesgue, let $s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k coskx + b_k sinkx)$, $\forall n \geq 1$ $with \quad s_0(f;x) = \frac{a_0}{2}$, denote the $(n+1)^{th}$ partial sums, called trigonometric polynomials of degree (or order) *n*, of the Fourier series of *f*: The conjugate series of the Fourier series of f is defined by $\sum_{n=1}^{\infty}(b_n cosnx - a_n sinnx) = \sum_{n=0}^{\infty} v_n$ and its n^{th} partial sum is defined as

$$\tilde{s}_n(f;x) = \sum_{k=1}^{n}(b_k coskx - a_k sinkx),$$
$$\forall n \geq 1 \ and \ \ \tilde{s}_n(f;x) = 0,$$

The conjugate of *f* denoted by $\tilde{f}$ is defined as

$$\tilde{f}(x) = -\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi} \psi_x(t)\cot\left(\frac{t}{2}\right)dt,$$

where $\psi_x(t) = f(x+t) - f(x-t)$, [[1], p. 131].
The L^p-norm of $f \in L^p[0, 2\pi]$ is defined by

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p dx\right\}^{\frac{1}{p}} (1 \leq p \leq \infty)$$
$$and \ \|f\|_\infty = \sup_{x\in[0,2\pi]}|f(x)|.$$

Let $\sum_{n=0}^{\infty} v_n$ be a given infinite series with its partial sums $\tilde{s}_n$. We denote by $\tilde{C}_n^{\alpha,\beta}$ the n^{th} Cesaro means of order (α,β), with $\alpha+\beta > -1$ of the sequence $\tilde{s}_n$, i.e. (see [4])

$$\tilde{C}_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta}\tilde{s}_v.$$

where $A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \alpha+\beta > -1$ and $A_0^{\alpha+\beta} = 1$. The series $\sum_{n=0}^{\infty} v_n$ is said to be summable $(\tilde{C},\alpha,\beta)$ to the definite number *s* if

$$\tilde{C}_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta}\tilde{s}_v \to s, \qquad as\ n \to \infty.$$

Furthermore, for *q*>0 a real number, the Euler means (E, *q*) of the sequence $\tilde{s}_n$ are defined [4] to be

$$\tilde{E}_n^q = \frac{1}{(1+q)^n}\sum_{v=0}^{n}\binom{n}{v} q^{n-v}\tilde{s}_v .$$

The series $\sum_{n=0}^{\infty} v_n$ is said to be summable $(\tilde{E}, q)$ to the definite number *s* if

$$\tilde{E}_n^q = \frac{1}{(1+q)^n}\sum_{v=0}^{n}\binom{n}{v} q^{n-v}\tilde{s}_v \to s,$$
$$as \quad n \to \infty.$$

The (E, *q*) transform of the (C, α,β) transform defines an (E, *q*) (C, α,β) transform and we shall denote it by $(EC)_n^{q,\alpha,\beta}$. Moreover, if

$$\widetilde{(EC)}_n^{q,\alpha,\beta} = \frac{1}{(1+q)^n}\sum_{v=0}^{n}\binom{n}{v} q^{n-v} C_k^{(\alpha,\beta)}\tilde{s}_k \to s,$$
$$as \quad n \to \infty.$$

then we shall say that the infinite series $\sum_{n=0}^{\infty} v_n$ is $(\tilde{E}, q)$ $(\tilde{C},\alpha,\beta)$ summable to the definite number *s*. We note that for q=1, $\alpha = 1\ and \beta = 0$ the concept of $(\tilde{E}, q)$ $(\tilde{C},\alpha,\beta)$ summability reduces to the $(\tilde{E}, 1)$ $(\tilde{C}, 1)$ summability introduced in [5].
We also write

$$K_n^{q,\alpha,\beta}(t) := \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{A_k^{\alpha+\beta}} \times \sum_{v=0}^{k}\frac{A_{k-v}^{\alpha-1}A_v^{\beta}\cos(v+1/2)t}{\sin t/2}$$

Known Results: Since the last two decades many researchers have been working on the degree of approximation$\tilde{f}$, conjugate of a function f belonging to $Lip\ \alpha, Lip\ (\alpha,p), Lip\ (\xi(t),p)$ and weighted Lipschitz class $W(L^p,\xi(t))$with $p \geq 1$ using different summability means of conjugate Fourier series of *f*.

Approximation of the conjugate to functions f belonging to $Lip\ \alpha$ and $Lip\ (\alpha,p)(p \geq 1)(0 < \alpha \leq 1)$ motivated the researchers to analyze the degree of approximation of the conjugate of the functions belonging to more general Lipschitz classes $Lip\ (\xi(t),p)$ and $W(L^p,\xi(t))(p \geq 1)$, where $\xi(t)$ is a positive increasing function. In the continuation of previous works Singh and Srivastava [10] have proved the degree of approximation of the conjugate of periodic functions through conjugate Fourier series by Hausdorff matrices, which represent a wider class of summability matrices, since (C,1), the Cesàro matrix of order 1, and (E, q), the Euler matrix of order q> 0, are Hausdorff matrices and the product of two Hausdorff matrices is also a Hausdorff matrix. Very recently, Krasniqi [11] has proved the degree of approximation of a 2π-periodic signal (function) *f*∈ $Lip\ \alpha$ means of Fourier series by (E, q) (C,α,β) or$(EC)_n^{q,\alpha,\beta}$. He proved:

Theorem 1: *If f is a 2π-periodic function that belongs to the Lip α class, then its degree of approximation is given by*

$$\left\|(EC)_n^{q,\alpha,\beta}(f) - f(x)\right\|_p = O((n+1)^{\alpha}),\quad 0 < \alpha < 1.$$

Remark 1:

Chandra [9] was the first to obtain the degree of approximation of *f*∈ $Lip\ (\alpha,p)$ as $\|t_n(f;x) - f(x)\|_p = O(n^{-\alpha})$ which is free from p; and the same was continued by Mittal et al. [8], Liendler [6] and Mittal et al. [7] for Fourier series. Many authors have proved the degree of approximation of the conjugate of periodic functions through conjugate Fourier series by different types of summability methods in various Lipschitz classes, which is dependent on p. Thus the degree of approximation of *f*∈ $W(L^p,\xi(t))$ has obtained in our theorems of this paper which is sharper than earlier results and free from *p*.

Main Results: In the problem of approximation, summability methods play an important role, particularly when the Fourier series of a function is not convergent. As mentioned in [2], the Lp-space in general, and L_2 and L_∞, in particular play an important role in the theory of signals and filters. We know that any series conjugate to a Fourier series is not necessarily itself a Fourier series [3], since series $\sum_{n=0}^{\infty}\frac{\cos nx}{\log(n+2)}$, is a Fourier series, but the corresponding sine series is not a Fourier series. Hence we need a separate study of conjugate Fourier series. Recently, Krasniqi [11] has proved the degree of approximation of a 2π periodic signal (function) *f*∈ $Lip\ \alpha$ means of Fourier series by (E, q) (C,α,β). The above mentioned observations motivate us to study further the problem of the degree of approximation of 2π periodic conjugate functions of f belonging to weighted Lipschitz class $W(L^p,\xi(t),(\gamma \geq 0))$ by (E, q)(C,α,β) means of their conjugate trigonometric Fourier series. Also, we minimize the error in degree of approximation as mentioned in remark 1 and prove:

Theorem 2: *Let f be a 2π-periodic function belonging to the weighted Lipschitz class* $W(L^p,\xi(t))$ *with p >1 and* $0 \leq \gamma \leq 1 - 1/p$. *Then the degree of approximation of* $\tilde{f}$*by (E, q)(C,*α,β*) means of conjugate Fourier series of f generated by* $(EC)_n^{q,\alpha,\beta}$, *is given by*

$$\left\|\widetilde{(EC)}_n^{q,\alpha,\beta}(f) - \tilde{f}(x)\right\|_p = O\left((n+1)^{\gamma}\xi\left(\frac{1}{(n+1)}\right)\right),\quad 0 < \alpha < 1.$$

provided a positive increasing function $\xi(t)$ *satisfies the following conditions:*

$\xi(t)/t$ *is non-increasing,*

$$\left(\int_0^{\frac{\pi}{n+1}}\left(|\psi_x(t)|\, sin^{\gamma}\frac{\frac{t}{2}}{\xi(t)}\right)^p dt\right)^{\frac{1}{p}} = O\left((n+1)^{-1/p}\right).$$

$$\left(\int_0^{\frac{\pi}{n+1}}\left(\frac{\xi(t)}{t\, sin^{\gamma}\left(\frac{t}{2}\right)}\right)^q dt\right)^{\frac{1}{q}} = O\left((n+1)^{\gamma+\frac{1}{p}}\xi(\pi/(n+1))\right),$$

$$\left(\int_{\frac{0\pi}{n+1}}^{\pi}\left(t^{-\delta}|\psi_x(t)|\, sin^{\gamma}\frac{\frac{t}{2}}{\xi(t)}\right)^p dt\right)^{\frac{1}{p}} = O\left((n+1)^{\delta-1/p}\right),$$

where δ *is an arbitrary number such that* $0 < \delta < \gamma + p^{-1}$ *and* $p^{-1} + q^{-1} = 1$ *for p>. The conditions (5) and (7) hold uniformly in x.*

The conditions (5) and (7) can be verified by using the fact that $\psi_x(t) \in W(L^p,\xi(t))$ and $\psi_x(t)/\xi(t)$

is a bounded function. The condition (6) is obvious in the light of the mean value theorem for integrals.
For the proof of our theorem, we need the following lemma:

Lemma 1: The estimate $K_n^{q,\alpha,\beta}(t) = O(t^{-1})$ holds true $0 < t \le \pi$.

Proof: Using $\sin(t/2) \ge t/\pi$ for $0 < t \le \pi$, we have $\sum_{v=0}^{k} A_{k-v}^{\alpha-1} A_v^{\beta} = A_k^{\alpha+\beta}$ and $\sum_{k=0}^{n} \binom{n}{k} q^{n-k} = (1+q)^n$, we have

$$\left|K_n^{q,\alpha,\beta}(t)\right| = \frac{1}{2\pi(1+q)^n}$$

$$\times \left|\sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{A_k^{\alpha+\beta}} \sum_{v=0}^{k} \frac{A_{k-v}^{\alpha-1} A_v^{\beta} \cos(v+1/2)t}{\sin t/2}\right|$$

$$\le \frac{1}{2t(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{A_k^{\alpha+\beta}} \left|\sum_{v=0}^{k} A_{k-v}^{\alpha-1} A_v^{\beta} e^{ivt}\right|$$

$$\le \frac{1}{2t(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} e^{ikt}$$

$$\le \frac{1}{2t(1+q)^n} (q+e^{it})^n = O(t^{-1}).$$

Proof of Theorem 2:

The integral representation of $\tilde{s}_n(f;x)$ is given by

$$\tilde{s}_n(f;x) - \tilde{f}(x)$$

$$= -\frac{1}{2\pi} \int_0^{\pi} \psi_x(t) \frac{\cos\left(\mathrm{k}+\frac{1}{2}\right)\mathrm{t}}{\sin\left(\frac{\mathrm{t}}{2}\right)} dt$$

Denoting $(EC)_n^{q,\alpha,\beta}(f)$ means of$\{\tilde{s}_n(f;x)\}$ by$\widetilde{(EC)}_n^{q,\alpha,\beta}(f)$, we write

$$\widetilde{(EC)}_n^{q,\alpha,\beta}(f) - \tilde{f}(x)$$

$$= \frac{1}{2\pi} \int_0^{\pi} \frac{\psi_x(t)}{\sin\left(\frac{\mathrm{t}}{2}\right)} \sum_{k=0}^{n} (EC)_n^{q,\alpha,\beta}(f) \cos\left(\mathrm{k}+\frac{1}{2}\right)\mathrm{t}\, dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} \frac{\psi_x(t)}{\sin\left(\frac{\mathrm{t}}{2}\right)} \frac{1}{(1+q)^n}$$

$$\times \sum_{k=0}^{n} \binom{n}{k} q^{n-k} C_k^{(\alpha,\beta)} \cos\left(\mathrm{k}+\frac{1}{2}\right)\mathrm{t}\, dt$$

$$= \int_0^{\pi} \psi_x(t) K_n^{q,\alpha,\beta}(t) dt$$

$$= \int_0^{\frac{\pi}{n+1}} \psi_x(t) K_n^{q,\alpha,\beta}(t) dt$$

$$+ \int_{\pi/(n+1)}^{\pi} \psi_x(t) K_n^{q,\alpha,\beta}(t) dt$$

$= I_1 + I_2$, say.

Using Holder's inequality, $\psi_x(t) \in W(L^p, \xi(t))$,condition (5), (6) and Lemma 1, we have,

$$|I_1| \le \left[\int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)| \sin^{\gamma}\left(\frac{t}{2}\right)}{\xi(t)}\right)^p dt\right]^{1/p}$$

$$\times \left[\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi/(n+1)} \left(\frac{\xi(t)\left|K_n^{q,\alpha,\beta}\right|}{\sin^{\gamma}\left(\frac{t}{2}\right)}\right)^q dt\right]^q$$

$$= O((n+1)^{\gamma})\, \xi(1/(n+1)).$$

Using Lemma 1, we have

$$|I_2| = \int_{\pi/(n+1)}^{\pi} \left|\frac{\psi_x(t)}{t}\right| dt.$$

Using Holder's inequality, $\psi_x(t) \in W(L^p, \xi(t))$,condition (4) and (7), we have

$$|I_2| \le \left[\int_{\frac{\pi}{n+1}}^{\pi} \left|t^{-\delta} |\psi_x(t)| \sin^{\gamma}(\frac{t}{2})/\xi(t)\right|^p dt\right]^{\frac{1}{p}}$$

$$\times \left[\int_{\pi/(n+1)}^{\pi} \left|\xi(t)/(t^{-\delta+1} \sin^{\gamma}(t/2))\right|^q dt\right]^{1/q}$$

$$= O\left((n+1)^{\delta-\frac{1}{p}+1}\right) \xi\left(\frac{\pi}{n+1}\right)$$

$$\times (n+1)^{(-\delta+\gamma-1/q)} \left[\int_{\pi/(n+1)}^{\pi} \left|t^{\delta-\gamma}\right|^q dt\right]^{1/q}$$

$$= O((n+1)^{\gamma} \xi(\pi/(n+1)),$$

in view of $p^{-1} + q^{-1} = 1$.

Collecting (8) - (10), we have

$$\left|\widetilde{(EC)}_n^{q,\alpha,\beta}(f) - \tilde{f}(x)\right|$$

$$= O((n+1)^{\gamma} \xi\left(\frac{1}{n+1}\right).$$

Hence,

$$\left\|\widetilde{(EC)}_n^{q,\alpha,\beta}(f) - \tilde{f}(x)\right\|_p$$

$$= \left(\frac{1}{2\pi} \int_0^{2\pi} \left|\widetilde{(EC)}_n^{q,\alpha,\beta}(f) - \tilde{f}(x)\right|^p dx\right)^{1/p}$$

$$= O((n+1)^{\gamma} \xi\left(\frac{1}{n+1}\right).$$

This completes the proof of Theorem 2.

Remark 2: Most of the authors mentioned above have taken $p \ge 1$ in their theorems and applied Holder's inequality without using L_{∞}- norm when $q = \infty$ (i.e., p = 1). Thus proofs of their theorems are not valid for p = 1. Therefore, we have taken p > 1 in our Theorem 2 stated above [3, pp.32-33].

For p = 1, we have the following result:

Theorem 3: Let f be a 2π periodic function belonging to the weighted Lipschitz class $W(L^1, \xi(t))$ with $0 \le \gamma < 1$. Then the degree of approximation of $\tilde{f}$ by (E, q)(C, α, β) means of conjugate Fourier series of f is given by

$$\left\|\widetilde{(EC)}_n^{q,\alpha,\beta}(f) - \tilde{f}(x)\right\|_1$$

$$= O\left((n+1)^{\gamma}\xi\left(\frac{1}{(n+1)}\right)\right), \quad 0<\alpha<1.$$

provided a positive increasing function $\xi(t)$ satisfies (4) and the following conditions:

$\xi(t)/t^{\gamma+\sigma}$ is non-decreasing,

$$\int_0^{\pi/(n+1)}\left(\frac{t^{\sigma-1}|\psi_x(t)|\sin^{\gamma}\left(\frac{t}{2}\right)}{\xi(t)}\right)dt = (n+1)^{-\sigma},$$

for some $\sigma>0$ such that $\sigma+\gamma<1$,

$$\int_{\frac{\pi}{n+1}}^{\pi}\frac{t^{-\delta}|\psi_x(t)|}{\xi(t)}dt = (n+1)^{\delta},$$

$\xi(t)/t^{\gamma-\delta+2}$ is non-increasing,

where $0<\delta<\gamma+1$. The conditions (12) and (13) hold uniformly in x.

Proof of Therorem 3:

Following the proof of Theorem 2 by (8), for p=1, i.e., $q=\infty$, we have $I_1=\int_0^{\pi/(n+1)}\psi_x(t)K_n^{q,\alpha,\beta}(t)dt$

Using $\psi_x(t)\in W(L^p,\xi(t))$, condition (11), (12) and Lemma 1, we have

$$|I_1| = \int_0^{\pi/(n+1)}\left(\frac{t^{\sigma-1}|\psi_x(t)|\sin^{\gamma}\left(\frac{t}{2}\right)}{\xi(t)}\right)dt$$

$$\times ess\sup_{0<t\leq\pi/(n+1)}\left|\frac{\xi(t)}{t^{\sigma}\sin^{\gamma}\left(\frac{t}{2}\right)}\right|$$

$$= O((n+1)^{-\sigma})\left|\frac{\xi(\pi/(n+1))}{(\pi/(n+1))^{\sigma+\gamma}}\right|$$

$$= O((n+1)^{\gamma}\,\xi(1/n+1)).$$

Using Lemma 1, we have

$$|I_2| = \int_{\pi/(n+1)}^{\pi}\left|\frac{\psi_x(t)}{t}\right|dt.$$

Using Holder's inequality, $\psi_x(t)\in W(L^p,\xi(t))$, condition (13) and (14), we have

$$|I_2| = \int_p^{\frac{\pi}{n+1}}\left(\frac{t^{-\delta}|\psi_x(t)|\sin^{\gamma}(\frac{t}{2})}{\xi(t)}\right)dt$$

$$\times ess\sup_{\pi/(n+1)<t\leq\pi}\left|\frac{\xi(t)}{t^{-\delta+\gamma+1}}\right|$$

$$= O((n+1)^{\delta})ess\sup_{\pi/(n+1)<t\leq\pi}\left|\frac{\xi(t)}{t^{-\delta+\gamma+1}}\right|dt$$

$$= O\left((n+1)^{\gamma}\xi\left(\frac{\pi}{n+1}\right)\right)$$

in view of $p^{-1}+q^{-1}=1$.

Collecting (15) - (16) with (8), we have\

$$\left|\widetilde{(EC)}_n^{q,\alpha,\beta}(f)-\tilde{f}(x)\right| = O\left((n+1)^{\gamma}\xi\left(\frac{\pi}{n+1}\right)\right).$$

Hence,

$$\left\|\widetilde{(EC)}_n^{q,\alpha,\beta}(f)-\tilde{f}(x)\right\|_1 = O\left((n+1)^{\gamma}\xi\left(\frac{\pi}{n+1}\right)\right).$$

This completes the proof of Theorem 3.

4. Corollaries:

The following corollaries can be derived from Theorem 3.

Corollary 1: If $\gamma=0$ then for $f\in Lip(\xi(t),p)$ with $p\geq1$,

$$\left\|\widetilde{(EC)}_n^{q,\alpha,\beta}(f)-\tilde{f}(x)\right\|_p = O\left(\xi\left(\frac{1}{(n+1)}\right)\right),$$

Corollary 2. If $\gamma=0$, $\xi(t)=t^{\alpha}(0<\alpha\leq1)$, then for $f\in Lip(\alpha,p)$ $(\alpha>1/p)$.

$$\left\|\widetilde{(EC)}_n^{q,\alpha,\beta}(f)-\tilde{f}(x)\right\|_p = O((n+1)^{-\alpha})$$

Corollary 3. If $p\to\infty$ in corollary 2, then for $f\in Lip\alpha$ $(0<\alpha<1)$,

$$\left\|\widetilde{(EC)}_n^{q,\alpha,\beta}(f)-\tilde{f}(x)\right\|_{\infty} = O((n+1)^{-\alpha})$$

5. Conclusion; We give some direct consequences of the main results. The (E,q)(C, α, β) meanscan be reduced to the following means:

1. If $\beta=0$ $then$ $\widetilde{(EC)}_n^{q,\alpha,\beta}(f)\equiv\widetilde{(EC)}_n^{q,\alpha,0}(f)=(E,\widetilde{q)(C},\alpha)(f)$,

2. If $\alpha=1$ $then$ $\widetilde{(EC)}_n^{q,\alpha,\beta}(f)\equiv\widetilde{(EC)}_n^{q,1,\beta}(f)=(E,\widetilde{q)(C},1,\beta)(f)$,

3. If $\beta=0,q=1$ $then$ $\widetilde{(EC)}_n^{q,\alpha,\beta}(f)\equiv\widetilde{(EC)}_n^{1,\alpha,0}(f)=(E,\widetilde{1)(C},\alpha)(f)$

4. If $\alpha=1,q=1$ $then$ $\widetilde{(EC)}_n^{q,\alpha,\beta}(f)\equiv\widetilde{(EC)}_n^{1,1,\beta}(f) = (E,\widetilde{1)(C},1,\beta)(f)$,

5. If $\alpha=q=1,\beta=0$ $then$ $\widetilde{(EC)}_n^{q,\alpha,\beta}(f)\equiv\widetilde{(EC)}_n^{1,1,0}(f)=(E,\widetilde{1)(C},1)(f)$,

Denoting $\widetilde{(EC)}_n^{q,\alpha,0}(f;x),\widetilde{(EC)}_n^{q,1,\beta}(f;x),\widetilde{(EC)}_n^{1,\alpha,0}(f;x)$are means of $\tilde{s}_n$. Therefore from theorems 2 and 3 lots of corollaries can be derived.

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