MEAN SQUARE SUM LABELING OF SOME GRAPHS

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Abstract: In this paper, we introduce a new labeling called mean square sum labeling. A bijection f:$V(G)$→{0,1, ..., p-1} G is said to be a *mean square sum labeling* if the induced function f*:E(G)→N given by f*(uv) = $\left\lfloor\frac{[f(u)]^2+[f(v)]^2}{2}\right\rfloor$or $\left\lceil\frac{[f(u)]^2+[f(v)]^2}{2}\right\rceil$ for every uv $\in E(G)$ is injective. A graph which admits a mean square sum labeling is called a mean square sumgraph. In this paper we prove that Path, Comb, Star graph, Cycle, Bistar, Doublestar, G = K_2+mK_1, Ladder, $P_n \odot K_2$ and some more graphs are mean square sum graphs.

Keywords: Labeling, mean square sum labeling, mean square sum graph.

1. Introduction: We begin with simple, finite, connected and undirected graph. For standard terminology and
notations we follow Harary [1]. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a *vertex labeling* (an *edge labeling*). Several types of graph labeling and a detailed survey is available in [2].

S. Somasundaram and R. Ponraj [3] have introduced the notion of mean labeling of graphs. A graph G with p vertices and q edges is called *mean graph*if there is an injective function f from the vertices of G to {0, 1, ..., q} such that when each edge uv is labeled with $\frac{f(u)+f(v)}{2}$if f(u)+f(v) is even and with $\frac{f(u)+f(v)+1}{2}$ if f(u)+f(v) is odd, then the resulting edge labels are distinct.

V. Ajitha, S. Arumugam and K. A. Germina [4] have introduced the notion of square sum labeling. A (p, q) graph G is said to be square sum, if there exists a bijection f : V(G)→{0, 1, ..., p-1} such that the induced function f*: E(G)→N defined by f*(uv) = $[f(u)]^2 + [f(v)]^2$for every uv $\in E(G)$ is injective. K. A. Germina and Reena Sebastian [5], [6] have investigated many results on this concept.

In this paper, we introduce mean square sum labeling of graphs. Let G = $(V(G), E(G))$be a graph. A bijection f:$V(G)$→{0, 1, ..., p-1} is said to be a *mean square sum labeling* if the induced function f*: E(G)→N given by f*(uv) = $\left\lfloor\frac{[f(u)]^2+[f(v)]^2}{2}\right\rfloor$or $\left\lceil\frac{[f(u)]^2+[f(v)]^2}{2}\right\rceil$for every uv $\in E(G)$ is injective. Not every graph is mean square sum. For example, any complete graph K_n, where n ≥ 6 is not mean square sum . We are interested to study different classes of graphs, which are mean square sum. In this paper we establish that Path, Comb, Star graph, Cycle,Bistar, Doublestar, G = K_2+mK_1, Ladder, $P_n \odot K_2$ and some more graphs are mean square sum graphs.

A brief summary of definitions and other information which are necessary for the present investigation are given below.

Definition 1.1. Let G = $(V(G), E(G))$be a graph. A bijection f:$V(G)$→{0, 1, ..., p-1} G is said to be a *mean square sum labeling* if the induced function f*: E(G)→N given by f*(uv) = $\left\lfloor\frac{[f(u)]^2+[f(v)]^2}{2}\right\rfloor$or $\left\lceil\frac{[f(u)]^2+[f(v)]^2}{2}\right\rceil$for every uv $\in E(G)$ is injective.

Definition 1.2. A graph which satisfies the mean square sum labeling is called a *mean square sum* graph.

Definition 1.3.A graph G is said to be *complete,* if every pair of its distinct vertices are adjacent. A complete graph on p vertices is denoted by K_p

Definition 1.4. A *star* graph is a complete bigraph $K_{1,n}$.

Definition 1.5. The *Bistar* $B_{m,n}$ is the graph obtained from K_2 by joining m pendent edges to one end of K_2 and n pendent edges to the other end of K_2. The edge of K_2 is called the central edge of $B_{m,n}$ and the vertices of K_2 are called the central vertices of $B_{m,n}$.

Definition 1.6. The corona of two graphs G_1 and G_2 is the graph G = $G_1 \odot G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where i^{th} vertex of G_1 is adjacent to every vertices in the i^{th} copy of G_2.

Definition 1.7. The graph $P_n \odot K_1$ is called a *comb*.

Definition 1.8. The product $P_2 \times P_n$ is called a *ladder*, and it is denoted by L_n. The ladder graph L_n is a planar undirected graph with 2n vertices and 3n-2 edges.

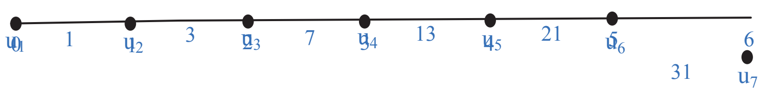
2. Main Results:

Theorem 2.1. Path P_n is a mean square sum graph.

Proof. Let G be the path P_n. Let V(G) = {$u_1, u_2, ..., u_n$} and E(G) = {u_iu_{i+1} / 1≤ i≤ n-1}. The mapping f : V(G) → {0, 1, ..., p-1} is defined by f(u_i) = i-1, 1≤ i≤ n and the induced function f*: E(G) → N is defined by f*(u_iu_{i+1}) = i^2-i+1, 1≤ i ≤ n-1 is injective. Hence the path P_n is a mean square sum graph.

Example 2.2. A mean square sum labeling of P_7 is given in figure 1.

Fig 1. P_7



Theorem 2.3. Comb $P_n \Theta K_1$ is a mean square sum graph.

Proof. Let P_n be the path $u_1u_2…u_n$. Let v_i be the vertex adjacent to u_i, $1 \le i \le n$. The resultant graph is $P_n \Theta K_1$. Let $G = P_n \Theta K_1$. Here $V(G) = \{u_i, v_i \ /1 \le i \le n\}$ and $E(G) = \{ u_iv_i, u_iu_{i+1}, u_nv_n \ /1 \le i \le n-1\}$. Then G has 2n vertices and 2n-1 edges. Define $f : V(G) \rightarrow \{0, 1, …, 2n-1\}$ by $f(u_i) = 2i-2$, $1 \le i \le n$; $f(v_i) = 2i-1$, $1 \le i \le n$. The induced function $f^*: E(G) \rightarrow N$ is defined by $f^*(u_iv_i) = 4i^2-6i+3$, $1 \le i \le n$; $f^*(u_iu_{i+1}) = 4i^2-4i+2$, $1 \le i \le n-1$ is injective. Hence Comb $P_n \Theta K_1$ is a mean square sum graph.

Example 2.4. A mean square sum labeling of $P_5 \Theta K_1$ is given in figure 2.

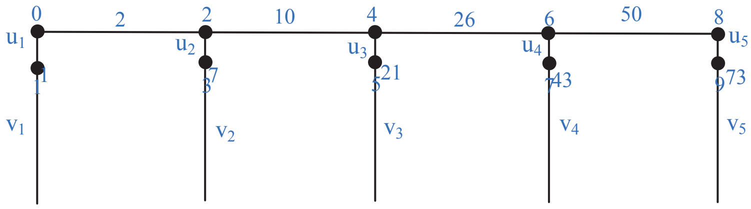


Fig 2. $P_5 \Theta K_1$

Theorem 2.5. The complete graph K_n is a mean square sum graph if and only if $n \le 5$.

Proof. Let G be the complete graph K_n with $V(K_n)=\{u_1, u_2, …, u_n\}$ and $E(K_n)=\{u_iu_j \ /1 \le i, j \le n \text{ and } i \ne j\}$. Define $f : V(K_n) \rightarrow \{0, 1, …, n-1\}$ by $f(u_i) = i-1$, $1 \le i \le n$. The induced function $f^*: E(K_n) \rightarrow N$ is defined by $f^*(uv) = \left\lfloor \frac{[f(u)]^2+[f(v)]^2}{2} \right\rfloor$ if $\{f(u), f(v)\} = \{0, 3\}$ otherwise $f^*(uv) = \left\lceil \frac{[f(u)]^2+[f(v)]^2}{2} \right\rceil$ is injective for $n \le 5$. In any labeling of vertices through 0, 1, …, n-1 of K_n, $n \ge 6$, there are three edges with end points labels 0, 5; 3, 4 and 1, 5 exist and $\frac{(0^2+5^2)}{2} = \frac{(3^2+4^2)}{2} = 12.5$ and $\frac{(5^2+1^2)}{2} = 13$. This implies that at least two edges get the same label which is a contradiction. Hence K_n is a mean square sum graph if and only if $n \le 5$.

Example 2.6. A mean square sum labeling of K_5 is given in figure 3.

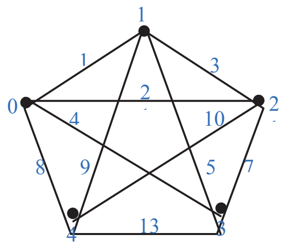


Fig 3. K_5

Theorem 2.7. The graph $G = K_2 + m K_1$ is a mean square sum graph.

Proof. Let $V(G) = \{v_1, v_2, u_i \ /1 \le i \le m\}$ where $V(K_2) = \{v_1, v_2\}$ and $E(G) = \{v_1v_2, v_1u_i, v_2u_i \ /1 \le i \le m\}$. Then G has m+2 vertices and 2m+1 edges. Define $f: V(G) \rightarrow \{0, 1, …, m-1\}$ by $f(u_i) = i+1$, $1 \le i \le m$; $f(v_j) = j-1$, $1 \le j \le 2$. The induced function $f^*: E(G) \rightarrow N$ is defined by $f^*(v_1v_2) = 1$; $f^*(v_1u_i) = \left\lfloor \frac{i^2+1}{2} + i \right\rfloor$, $1 \le i \le m$; $f^*(v_2u_i) = \left\lceil \frac{i^2}{2} + i + 1 \right\rceil$, $1 \le i \le m$ is injective. Hence it follows that f^* admits a mean square sum labeling. Hence G is a mean square sum graph.

Example 2.8. A mean square sum labeling of K_2+5K_1 is given in figure 4.

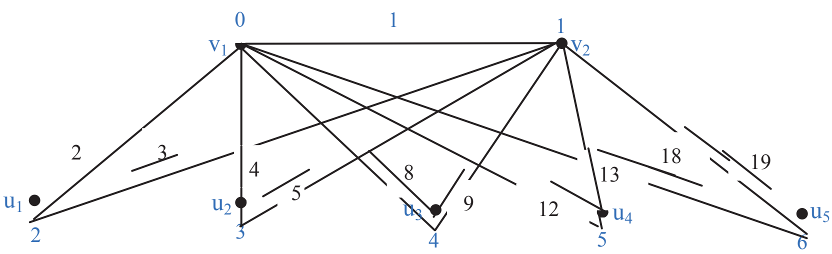


Fig 4. K_2+5K_1

Theorem 2.9. Cycle C_n is a mean square sum graph for $n \geq 3$.

Proof. Let G be the cycle C_n. Let V(G) = {u_i / 1≤ i≤ n} and E (G) = { u_iu_{i+1}, u_nu_1/ 1≤ i≤ n-1}. Then G has n vertices and n edges. Define f :V(G)→{0, 1, ..., n-1} by $f(u_i)$ = i-1, 1≤ i≤ n. The induced function f*: E(G)→N is defined by $f^*(u_iu_{i+1}) = i^2-i+1$, 1≤i≤ n-1 and $f^*(u_nu_1) = \left\lfloor \frac{(n-1)^2}{2} \right\rfloor$ is injective. Hence the cycle C_n is a mean square sum graph.

Example 2.10. A mean square sum labeling of C_6 is given in figure 5.

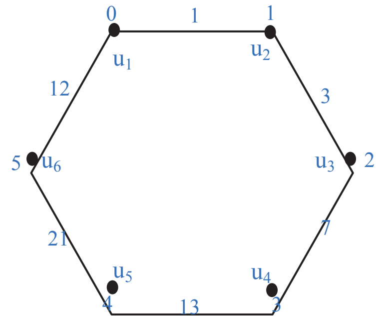


Fig 5. C_6

Theorem 2.11. The ladder $L_n = P_n \times K_2$ is a mean square sum graph.

Proof. Let $V(L_n)$ = {u_i,v_i /1≤ i≤ n} and $E(L_n)$ = {u_iu_{i+1}, v_iv_{i+1}, u_iv_i, u_nv_n /1≤ i≤ n-1}. Then G has 2n vertices and 3n-2 edges. Define f : $V(L_n)$→{0, 1, ..., 2n-1} by $f(u_i)$ = 2i-2, 1≤ i≤ n; $f(v_i)$ = 2i-1, 1≤ i≤ n. The induced function f*: E(G)→N is defined by $f^*(u_iv_i) = 4i^2-6i+3$, 1≤i≤n; $f^*(u_iu_{i+1}) = 4i^2-4i+2$, 1≤ i≤ n-1; $f^*(v_iv_{i+1}) = 4i^2+1$, 1≤ i≤ n-1 is injective. Hence the ladder L_n is a mean square sum graph.

Example 2.12. A mean square sum labeling of L_5 is given in figure 6.

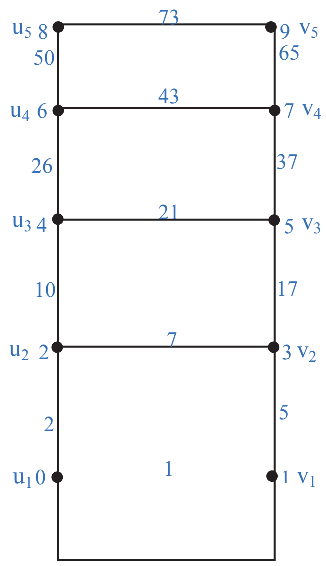


Fig 6. L_5

Theorem 2.13. $P_n \odot K_2$ is a mean square sum graph.

Proof. Let $u_1u_2....u_n$ be the path P_n and let v_i, w_i be the vertices of i^{th} copy of K_2 which are joined to the vertex u_i of path P_n, 1≤i≤n. The resultant graph is G = $P_n \odot K_2$ with V(G) = {u_i, v_i, w_i /1≤ i≤ n} and E(G) = {u_iu_{i+1}/1≤i≤n-1}∪ {u_iv_i, u_iw_i, v_iw_i /1≤i≤n}.Then G has 3n vertices and 4n-1 edges. Define f: V(G)→ {0, 1, ..., 3n-1} by $f(u_i)$ = 3i-3, $f(v_i)$ = 3i-2, and $f(w_i)$ = 3i-1, for 1≤i≤n. The induced function f*: E(G)→N is defined by $f^*(u_iw_i) = 9i^2-12i+5$, 1≤i≤n; $f^*(v_iw_i) = 9i^2-9i+3$, 1≤i≤n; $f^*(u_iv_i) = 9i^2-15i+7$, 1≤i≤n; $f^*(u_iu_{i+1}) = 9i^2-9i+5$, 1≤i≤n-1 is injective. Hence $P_n \odot K_2$ is a mean square sum graph.

Example 2.14. A mean square sum labeling of $P_4 \odot K_2$ is given in figure 7.

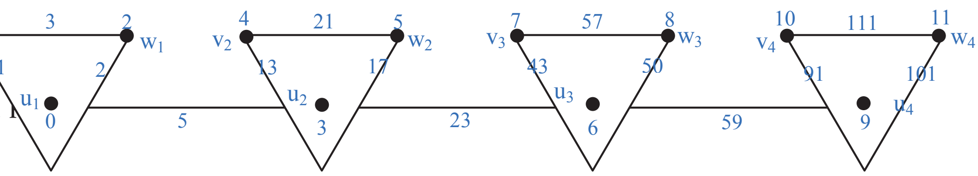


Fig 7.

Theorem 2.15. Agraph obtained by attaching the central vertex of $K_{1,2}$ at each pendent vertex of a comb $P_n \odot K_2$ is a mean square sum graph.

Proof. Let $u_1u_2 \ldots u_n$ be the path P_n and let v_i be a vertex adjacent to u_i, $1 \le i \le n$. The resultant graph is $P_n \odot K_1$. Let x_i, w_i, y_i be the vertices of i^{th} copy of $K_{1,2}$ with the central vertex w_i. Identify the vertex w_i with v_i, $1 \le i \le n$, we get the required graph is G.Then $V(G) = \{u_i, v_i, y_i, x_i \,/\, 1 \le i \le n\}$ and $E(G) = \{u_iv_i, v_ix_i, v_iy_i, u_ju_{j+1} \,/\, 1 \le i \le n, 1 \le j \le n-1\}$ and hence G has 4n vertices and 4n-1 edges. Define $f: V(G) \to \{0, 1, \ldots, 4n-1\}$ by $f(u_i) = 4i-4$, $f(v_i) = 4i-3$, $f(x_i) = 4i-2$, and $f(y_i) = 4i-1$, for $1 \le i \le n$. The induced function $f^*: E(G) \to N$ is defined by $f^*(u_iv_i) = 16i^2-28i+13$, $1 \le i \le n$; $f^*(v_ix_i) = 16i^2-20i+7$, $1 \le i \le n$; $f^*(v_iy_i) = 16i^2-16i+5$, $1 \le i \le n$; $f^*(u_ju_{j+1}) = 16j^2-16j+8$, $1 \le j \le n-1$ is an injective function. Hence G is a mean square sum graph.

Example 2.16. A mean square sum labeling of G when n = 4 is given in figure 8.

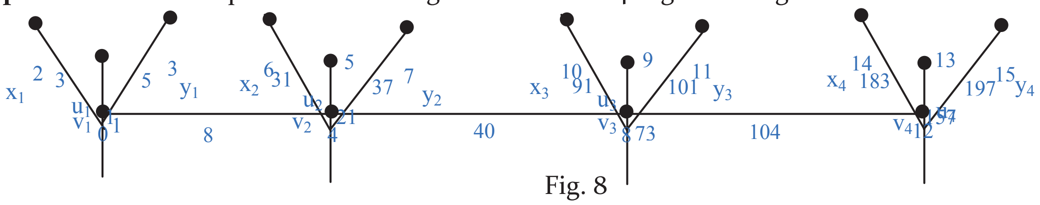


Fig. 8

Theorem 2.17. Agraph obtained by attaching a triangle K_3 at each pendent vertex of a comb $P_n \odot K_1$ is a mean square sum graph.

Proof. Let G be the graph obtained by attaching a triangle at each pendent vertex of a comb. Let P_n be the path $u_1u_2 \ldots u_n$. Let v_i be the vertex adjacent to u_i, $1 \le i \le n$. The resultant graph is $P_n \odot K_1$. Let x_i, z_i, y_i be the vertices of i^{th} copy of K_3. Identify the vertex z_i with v_i, $1 \le i \le n$, we get the required graph G with $V(G) = \{u_i, v_i, x_i, y_i \,/ 1 \le i \le n\}$ and $E(G) = \{u_iu_{i+1} / 1 \le i \le n-1\} \cup \{u_iv_i, v_ix_i, v_iy_i, y_ix_i \,/ 1 \le i \le n\}$. Then G has 4n vertices and 5n-1 edges. Define $f: V(G) \to \{0, 1, \ldots, 4n-1\}$ by $f(u_i) = 4i-4$, $f(v_i) = 4i-3$, $f(x_i) = 4i-2$, and $f(y_i) = 4i-1$, for $1 \le i \le n$. The induced function $f^*: E(G) \to N$ is defined by $f^*(u_iv_i) = 16i^2-28i+13$, $1 \le i \le n$; $f^*(v_ix_i) = 16i^2-20i+7$, $1 \le i \le n$; $f^*(v_iy_i) = 16i^2-16i+5$, $1 \le i \le n$; $f^*(u_iu_{i+1}) = 16i^2-16i+8$, $1 \le i \le n-1$; $f^*(y_ix_i) = 16i^2-12i+3$, $1 \le i \le n$ is injective. Hence G is a mean square sum graph.

Example 2.18. A mean square sum labeling of G when n = 4 is given in figure 9.

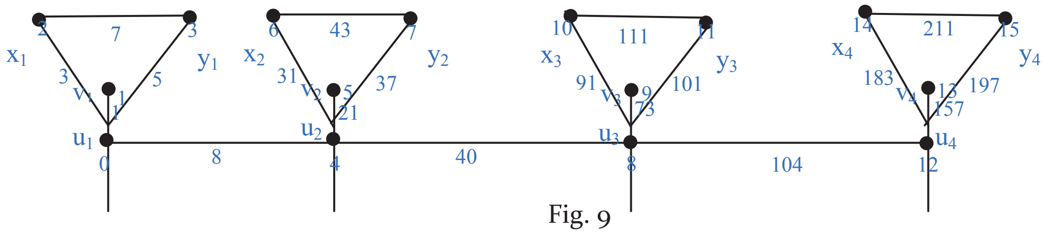


Fig. 9

Theorem 2.19. Star graph $K_{1,n}$ is a mean square sum graph.

Proof. Let G be the star graph $K_{1,n}$. Let $V(G) = \{u, u_i \,/\, 1 \le i \le n\}$ and $E(G) = \{ uu_i \,/\, 1 \le i \le n\}$. Then G has n+1 vertices and n edges. Define $f: V(G) \to \{0, 1, \ldots, n\}$ by $f(u) = 0$ and $f(u_i) = i$, $1 \le i \le n$. The induced function $f^*: E(G) \to N$ is defined by $f^*(uu_i) = \left\lceil \frac{i^2}{2} \right\rceil$, $1 \le i \le n$ is injective. Hence the Star graph $K_{1,n}$ admits a mean square sum labeling.

Example 2.20. A mean square sum labeling of $K_{1,6}$ is given in figure 10.

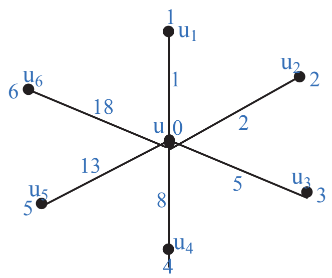


Fig 10. $K_{1,6}$

Theorem 2.21. The bistar $B_{m,n}$ is a mean square sum graph.

Proof. Let u, v be the vertices of K_2. Join m pendent vertices $u_1, u_2, \ldots, u_m$ to the one end of K_2 and join n pendent vertices $v_1, v_2, \ldots, v_n$ to the other end of K_2. The resultant graph G is the bistar $B_{m,n}$ with $V(G) = \{u, u_1, \ldots, u_m, v, v_1, \ldots, v_n\}$ and $E(G) = \{uu_i, vv_j, uv \ / \ 1\le i\le m, 1\le j\le n\}$. Then $B_{m,n}$ has m+n+2 vertices and m+n+1 edges.Define $f:V(G)\rightarrow\{0, 1, \ldots, m+n+1\}$ by f(u) = 0; f (v) = m+1; $f(u_i) = i$, $1\le i\le m$, $f(v_i) = m+i+1$, $1\le i\le n$. The induced function $f^*:E(G)\rightarrow N$ is defined by $f^*(uu_i) = \left\lceil \frac{i^2}{2} \right\rceil$, $1\le i\le m$; $f^*(vv_i) = \left\lceil (m+1)^2 + i(m+1) + \frac{i^2}{2} \right\rceil$, $1\le i\le n$; $f^*(uv) = \left\lceil \frac{(m+1)^2}{2} \right\rceil$ is injective. Hence bistar $B_{m,n}$ is a mean square sum graph.

Example 2.22.A mean square sum labeling of $B_{6,6}$ is given in figure 11.

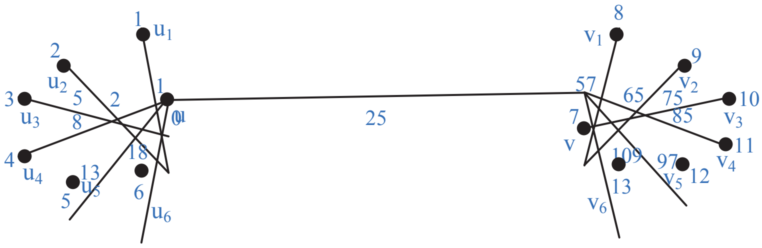


Fig 11. $B_{6,6}$

Theorem 2.23. The double star $K_{1,n,n}$ is a mean square sum graph.

Proof. Let G be the double star $K_{1,n,n}$. Let $V(G) = \{u, u_1, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be the vertices of the double star $K_{1,n,n}$ and $E(G) = \{uu_i, u_iv_i \ /1\le i\le n\}$. Then $K_{1,n,n}$ has 2n+1 vertices and 2n edges. Define $f:V(G)\rightarrow\{0, 1, \ldots, 2n\}$ by f(u) = 0; $f(u_i) = i$, $1\le i\le n$ and $f(v_i) = n+i$, $1\le i\le n$. The induced function $f^*:E(G)\rightarrow N$ is defined by $f^*(uu_i)=\left\lceil \frac{i^2}{2} \right\rceil$, $1\le i\le n$ and $f^*(u_iv_i)=\left\lceil i^2 + ni + \frac{n^2}{2} \right\rceil$, $1\le i\le n$ is injective. Hence double star $K_{1,n,n}$ is a mean square sum graph.

Example 2.24. A mean square sum labeling of $K_{1,7,7}$ is given in figure 12.

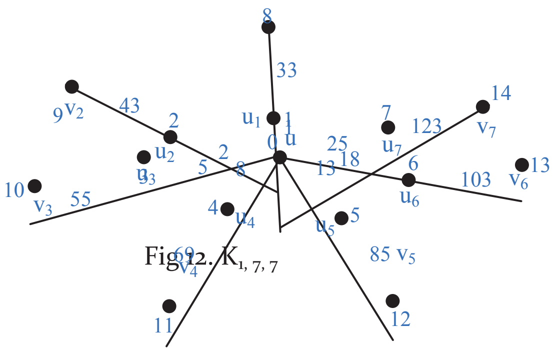


Fig12. $K_{1,7,7}$

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