DELTA GENERALIZED b-CLOSED FUNCTIONS IN TOPOLOGICAL SPACES

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Abstract: In this paper, the notion of δgb-closed functions, δgb^*-closed functions, almost δgb-closed functions, b-δgb-closed functions and regular δgb-closed functions in topological spaces are introduced. Some of their properties are studied and characterizations are established.

Keywords: b-closed, δgb-closed, δgb^*-closed, almost δgb-closed and b- δgb-closed.

Introduction: Weaker and stronger forms of closed functions have been introduced and studied by several topologists like E.Ekici[4] ,T.Noiri[6], M.Mrsevic and et.al [5] who have introduced and studied b-closed functions δ-closed functions and δ-open functions in topological spaces.

Throughout this paper, (X,τ),(Y,σ) and (Z,η) (or simply X,Y and Z) represents topological spaces on which no separation axioms are assumed unless explicitly stated. The b-closure of a subset A of X is the intersection of all b-closed sets containing A and is denoted by bcl(A).

2.Preliminaries: The following definitions and results are useful in the sequel,

Definition 2.1 A subset A of a topological space X is called,

(i) b-closed[1] if cl(int(A))∩int(cl(A))⊆A.

(ii)regular-closed [7] if A=cl(int(A)).

(iii)δ-closed[8] if A=clδ(A) where clδ(A)={x∈X:int(cl(U)∩A≠Φ,U∈τ and x∈U}.

(iv) delta generalized b-closed (briefly, δgb-closed)[2] if bcl(A)⊆ G whenever A⊆ G and G is δ-open in X.

The complements of the above mentioned closed sets are their respective open sets.

Definition 2.2 A function f:X→Y from a topological space X into a topological space Y is called,

(i) γ-closed(=b-closed)[4] if f(G) is b-closed in Y for every closed set G of X.

(ii) strongly γ-closed(=strongly b-closed)[4] if f(G) is b-closed in Y for every b-closed set G of X.

(iii) quasi γ-closed(=quasi b-closed)[4] if f(G) is closed in Y for every b-closed set G of X.

(iv) gγ-closed(= gb closed) [4] if f(G) is gb-closed in Y for every closed set Gof X.

(v) bgγ-closed(=b-gb-closed)[4]if f(G) is gb-closed in Y for every b-closed set G of X.

(vi) δ-closed[6]if f(G) is δ-closed in Y for every δ-closed set G of X.

(vii) δ-open[5]if f(G) is δ-open in Y for every δ-open set G of X.

(viii) almost gγ-closed(=almost gb-closed)[4] if f(G) is gb-closed in Y for every regular closed set G of X.

(ix) δgb-continuous[3] if f^{-1}(V)is gb-closed in X for each closed set V in Y .

(x) pre- δgb-continuous[3] if f^{-1}(V) is gb-closed in X for each b-closed set V in Y.

(xi) δgb-irresolute[3] if f^{-1}(V) is gb-closed in X if for each gb-closed V in Y.

Definition 2.3 [3]. A topological space X is said to be,

(i) $T_{\delta gb}$-space if every δgb-closed subset of X is closed.

(ii) $\delta gbT_{1/2}$ -space if every δgb-closed subset of X is b-closed.

(iii) $\delta T_{\delta gb}$-space if every δgb-closed subset of X is δ-closed.

(iv) $rT_{\delta gb}$-space if every δgb-closed subset of X is regular closed.

3.Delta Generalized b-Closed Functions

In this section, the concepts of δgb-closed functions, δgb^*-closed functions, almost δgb-closed functions and b-δgb-closed functions in topological spaces are introduced. Some of their properties are studied and characterizations are established.

Definition 3.1. A function f:X→Y is said to be,

(i) δgb-closed if f(A) is δgb-closed in Y for each closed set A of X.

(ii) δgb^*-closed if f(A) is δgb-closed in Y for each -closed set A of X.

(iii) b-δgb-closed if f(A) is δgb-closed in Y for each b-closed set A of X.

(iv) almost δgb-closed if f(A) is δgb-closed in Y for each regular-closed set A of X.

The complements of the above mentioned closed functions are their respective open functions.

From the above definitions, The following implications holds and none of its implications is reversible.

Quasi b-closed→b-δgb-closed→δgb-closed→δgb^*-closed→almost δgb-closed.

Example 3.2 Let X=Y={a,b,c,d} and τ=σ={X,Φ,{a},{b},{a,b},{a,c}},{a,b,c}}.

Let f:X→Y be a function defined as follows: f(a)=d=f(d),f(b)=b and f(c)=a.

Then f is δgb-closed but not b-δgb-closed, since {b,c} is b-closed in X but f({b,c})={a,b} is not δgb-closed in Y.

If g:X→Y is a function defined as follows: g(a)=d=g(b),g(c)=a and g(d)=b.Then g is δgb^*-closed but not δgb-closed, since {c,d} is closed in X but g({c,d})={a,b} is not δgb-closed in Y.

If h:X→Y is a function defined as follows: h(a)=b,h(b)=a,h(c)=c and h(d)=d.
Then h is b-δgb-closed but not quasi b-closed, since {b} is b-closed in X
But h({b})={a} is not closed in Y.

Example3.3 Let X=Y={a,b,c,d} and τ=σ={X,Φ,{a},{b},{a,b}}. Let f:X→Y be a function defined as follows: f(a)=c,f(b)=d and f(c)=a and f(d)=b.Then f is almost δgb-closed but not δgb^*-closed, since {c,d} is δ-closed in X but f({c,d})={a,b} bg is not δgb-closed in Y.

Theorem3.4 : Let f:X→Y be a function. Then ,
(i) If f is strongly b-closed, then it is b-δgb- closed.
(ii) If f is b-closed(resp.closed), then it is δgb-closed.
(iii) If f is δ-closed,then it is δgb^*-closed.
(iv) If f is gb-closed, then it is δgb-closed.
(v) If f is b-gb-closed, then it is b-δgb- closed.
(vi) If f is almost gb-closed then it is almost δgb-closed.

Proof: Follows from definitions.

Remark 3.5. The converse of the theorem 3.4 need not be true as seen from the following examples.

Example 3.6. In example 3.2, f is δgb-closed but not b-closed.

Example 3.7. In example 3.2 , g is δgb^*-closed but not δ-closed.

Example 3.8. In example 3.2, h is b-δgb-closed but not b-gb-closed and strongly b-closed.

Example 3.9. Let X=Y={a,b,c,d} and τ=σ={X,Φ,{a},{b},{a,b},{a,c},{a,b,c}} .
Let f:X→Y be a function defined as follows:f(a)=c,f(b)=a,f(c)=d and f(d)=a.
Then f is almost δgb-closed but not almost gb-closed , since {b,d} is regular-closed in X but f({b,d})={a} is not gb-closed in Y.
If g:X→Y is a function defined as follows: g(a)=c,g(b)=d,g(c)=c and g(d)=a.
Then g is δgb-closed but not gb-closed, since {d} is closed in X but g({d})={a} is not gb-closed in Y.

Theorem 3.10. Let f:X→Y be a function.
(i) If f is δgb-closed and Y is $T_{\delta gb}$-space, then f is closed.
(ii) If f is b-δgb-closed and Y is $\delta gbT_{1/2}$- space, then f is strongly b-closed.
(iii) If f is b-δgb-closed and Y is $T_{\delta gb}$-space, then f is quasi b-closed.
(iv) If f is δgb^*-closed and Y is $\delta T_{\delta gb}$-space, then it is δ-closed.

Remark 3.11 The following examples show that b-δgb-closed and gb-closed functions are independent.

Example 3.12. In example 3.9, g is b-δgb-closed but not gb-closed.

Example 3.13. In example 3.2,f is gb-closed but not b-δgb-closed.

Lemma 3.14 .A function f:X→Y is b-δgb-closed if and only if for each subset B of Y and each b-open set U containing f^{-1}(B),there exists a δgb-open set G of Y such that B⊂G and f^{-1}(G)⊂U.

Theorem 3.15 [2]. A subset A of a topological space X is δgb-open if and only if M⊂bint(A) whenever M is δ-closed and M⊂A.

Corollary 3.16. If f:X→Y is b-δgb-closed,then each δ-closed set K of Y and each b-open set U containing f^{-1}(K),there exists a b-open set V containing K such that f^{-1}(V)⊂U.

Proof: Suppose that f:X→Y is b-δgb-closed.
Let K be any δ-closed set of Y and U be any b-closed set containing f^{-1}(K).By Lemma 3.14, there exists a δgb-open set H of Y such that K⊂H and f^{-1}(H)⊂U.
Since K is δ-closed,then by Theorem 3.15, K⊂bint(H).
Put bint(H)=V, then V is a b-open set such that K⊂V and f^{-1}(V)⊂ U.

Definition 3.17 [3]. Let A be a subset of a space X.Then the δgb closure of A is defined to be the intersection of all δgb-closed sets containing A and it is denoted by δgbcl(A).

Theorem 3.18. Let A and B be subsets of a topological space X.Then
(i) δgbcl(X)=X and δgbcl(Φ)=Φ.
(ii) If A⊂B, then δgbcl(A)⊂δgbcl(B).
(iii)δgbcl(A)∪gbcl(B)⊂gbcl(A∪B).
(iv) δgbcl(A∩B)⊂δgbcl(A)∩δgbcl(B).
(v) A is δgb closed if and only if δgbcl(A)=A.
(vi) δgbcl(δgbcl(A))=δgbcl(A).
(vii) A⊂δgbcl(A)⊂bcl(A).

Proof: The easy verification is omitted.

Remark 3.19. The equalities do not hold in results (iii) and(iv) as seen from the following examples.

Example 3.20. Consider X={a,b,c,d }with the topology τ={X, Φ,{b},{c},{a, b},{b,c},{a,b,c},{b,c,d}}.
(iii) If A={b}and B={c}Then δgbCl(A)={b}, δgbCl(B)={c} and δgbCl(A∪B)={b,c,d}.
Thus we have δgbCl(A∪B)={b,c,d} ≠{b,c}=δgbCl(A)∪δgbCl(B) .
(iv) If A={b,c} and B={b,d} .Then δgbCl(A)={b,c,d},δgbCl(B)={b,d} and δgbCl(A∩B)={b}. Thus we have δgbCl(A)∩δgbCl(B)={b,d}≠{b}=δgbCl(A∩B).

Theorem 3.21.A function f:X→Y is b-δgb-closed if and only if δgbcl(f(A))⊂f(bcl(A)), for every subset A of X.

Proof: Suppose that f:X→Y is b-δgb-closed and A⊂X.Then f(bcl(A)) is δgb-closed in Y. Since f(A)⊂f(bcl(A)), then gbcl(f(A)⊂δgbcl(f(bcl(A)))=f(bcl(A)).
Conversely, let A be any b-closed set in X. Then A=bcl(A),implies f(A)=f(bcl(A)).
By hypothesis, δgbcl(f(A)⊂f(bcl(A)) =f(A). But

f(A)⊂gbcl(f(A)),so f(A)=gbcl(f(A)).Therefore f is δgb-closed in Y and hence f is b-δgb-closed.

Remark 3.22. The composition of two b-δgb-closed (resp, δgb-closed, δgb^*-closed, almost δgb-closed) functions need not be b-δgb-closed (resp,δgb-closed,δgb^*-closed, almost δgb-closed) in general as seen from the following examples.

Example3.23. Let X=Y=Z={a,b,c}. Let τ={X,Φ,{a},{b,c}},σ=={Y,Φ,{a}} and η={Z,Φ,{a},{b},{a,b}} be topologies on X,Y and Z respectively Let f:X→Y be a function defined by f(a)=c,f(b)=a and f(c)=b and g:Y→Z be a function defined by g(a)=a,g(b)=b and g(c)=c. Then f and g are b-δgb-closed (resp, δgb-closed, δgb^*-closed,almost δgb-closed) but (g_*f):X→Z is not b-δgb-closed (resp,δgb-closed, δgb^*-closed, almost δgb-closed), since there exists a b-closed ((resp,closed,δ-closed, regular-closed) set {b,c} in X such that (g_*f)({b,c})={a,b} is not δgb-closed in Z.

Theorem 3.24. Let f:X→Y and g:Y→Z be any two functions.

(i) If f is strongly b-closed and g is b-δgb- closed, then g_*f is b-δgb-closed.

(ii) If f is b-closed and g is b-δgb- closed, then g_*f is δgb-closed.

(iii) If f is quasi b-closed and g is δgb- closed, then g_*f is b-δgb-closed.

(iv) If f is δ-closed and g is δgb- closed,then g_*f is δgb-closed.

(v) If f is closed and g is δgb-closed, then g_*f is δgb-closed.

Proof. (i) Let h= g_*f and A be a b-closed set in X. Since f is strongly b-closed , f(A) is b-closed in Y. Therefore g[f(A)]=h(A) is δgb-closed in Z because g is b-δgb-closed. Hence g_*f is b-δgb-closed.

The proofs of (ii),(iii),(iv) and (v) are analogous to (i) with obvious changes.

Theorem 3.25. Let f:X→Y and g:Y→Z be any two functions.

(i) If Y is $T_{\delta gb}$-space ,f and g are δgb- closed, then g_*f is δgb-closed.

(ii) If Y is $\delta T_{\delta gb}$-space ,f and g are δgb^*- closed, then g_*f is δgb^*-closed.

(ii) If Y is $T_{\delta gb}$-space ,f is δgb^*-closed and g is δgb-closed ,then g_*f is δgb^*-closed.

(iv) If Y is $\delta T_{\delta gb}$-space, f is almost δgb- closed and g is δgb^*-closed, then g_*f is almost δgb-closed .

(v) If Y is $\delta gbT_{1/2}$-space ,f and g are b-δgb- closed, then g_*f is b-δgb-closed.

(vi) If Y is $rT_{\delta gb}$-space ,f and g are almost δgb-closed, then g_*f is almost δgb-closed.

Proof: (i) Let h=g f and A be a closed set in X.Since f is δgb-closed, f(A) is δgb-closed in Y.Since Y is $T_{\delta gb}$-space, f(A) is closed in Y.

Since g is δgb-closed, g[f(A)]=h(A) is δgb-closed in Z and hence g_*f is δgb-closed.

The Proofs of (ii),(iii),(iv),(v) and (vi) are analogous to (i) with obvious changes.

Theorem 3.26 . Let f:X→Y be a function.

(i) If Y is $T_{\delta gb}$-space then f is δgb-closed if and only if it is closed.

(ii) If Y is $\delta gbT_{1/2}$-space, then f is δgb- closed if and only if it is b-closed.

(iii) If Y is $\delta gbT_{1/2}$-space,then f is b-δgb-closed if and only if it is strongly b-closed.

(iv) If Y is $T_{\delta gb}$-space,then f is b-δgb- closed if and only if it is quasi b- closed.

(v) If Y is $\delta T_{\delta gb}$-space ,then f is δgb^*- closed if and only if it is δ-closed.

Proof:(i) Suppose Y is $T_{\delta gb}$-space and f is δgb-closed. Let A be a closed set in X, then f(A) is δgb-closed in Y and hence f(A) is closed in Y. Therefore f is closed.

Converse is obvious,since every closed set is δgb-closed.

(ii) Suppose Y is $\delta gbT_{1/2}$-space and f is δgb-closed. Let A be a closed set in X, then f(A) is δgb-closed in Y and hence f(A) is b-closed in Y.

(ii) Suppose Y is $\delta gbT_{1/2}$-space and f is δgb-closed. Let A be a closed set in X, then f(A) is δgb-closed in Y and hence f(A) is b-closed in Y.

Converse is obvious. since every b-closed set is δgb-closed

(iii) Suppose Y is $\delta gbT_{1/2}$-space and f is b-δgb-closed. Let A be a b-closed set in X, then f(A) is δgb-closed in Y and hence f(A) is b-closed in Y. Converse is obvious.

(iv) Suppose Y is $T_{\delta gb}$-space and f is b-δgb-closed.Let G be a b-closed set in X, then f(G) is δgb-closed in Y and hence f(G) is closed in Y. Converse is obvious.

(v) Suppose Y is $\delta T_{\delta gb}$-space and f is δgb^*-closed.Let G be a δ-closed set in X, then f(G) is δgb-closed in Y and hence f(G) is δ-closed in Y.

Converse is obvious.

Theorem 3.27.For any bijection f:X→Y,the following are equivalent.

(i) inverse of f is pre-δgb- continuous

(ii) f is a b-δgb-closed function.

(iii) f is a b-δgb-open function.

Proof:(i)→(ii):Let M be a b-closed set in X,then by (i), $(\mathbf{f}^{-1})^{-1}$(M)=f(M) is δgb-closed in Y and hence f is b-δgb-closed.

(ii)→(iii):Let F be a b-open set in X,then X-F is a b-closed set in X.By (ii) f(X-F)=Y-f(F) is δgb-closed in Y, implies f(F) is δgb-open in Y.

(iii)→(i):Let N be a b-open set in X.By (iii),f(N)=$(\mathbf{f}^{-1})^{-1}$(N) is δgb-open in Y.

Therefore $\mathbf{f}^{-1}$ is pre-δgb-continuous.

Theorem 3.28 [3]. If the bijective function f:X→Y is pre-δgb-continuous and open, then f is δgb-irresolute.

Theorem 3.29. Let f:X→Y and g:Y→Z be any two functions such that (g∗f):X→Z is b-δgb-closed, then following results hold.

(i) If f is b-continuous(resp,b-irresolute) and surjective,then g is b-δgb-closed function.

(iii) If g is δgb-irresolute and injective , then f is b-δgb-closed function.

(iv) If g is δ-open and pre-δgb-continuous bijective ,then f is b-δgb-closed function.

Proof:(i)Suppose A is a closed(resp,b-closed) in Y. Since f is b-continuous (resp,b-irresolute), then f^{-1}(A) is b-closed in X. Since (g∗f) is b-δgb-closed and f is surjective, then (g∗f)(f^{-1}(A))=g(A) is gb-closed in Z. Therefore g is b-δ gb-closed.

(ii)Suppose K be a b-closed in X. Since (g∗f) is b-δgb-closed, then (g∗f)(K) is δgb-closed in Z. Since g is δgb-irresolute and injective, then g^{-1}(g∗f)(K)=f(K) is δgb-closed in Y. Therefore f is pre-δgb-closed.

(iii) Suppose G be a b-closed in X.Since (g∗f) is pre-δgb-closed,then (g∗f)(G) is δgb-closed in Z. By Theorem 3.28, g^{-1}(g∗f)(G)=f(G) is δgb-closed in Y. Therefore f is b-δgb-closed.

Lemma 3.30 .A function f:X→Y is almost δgb-closed if and only if for each subset B of Y and each regular-open set U containing f^{-1}(B),there exists a δgb-open set G of Y such that B⊂G and f^{-1}(G)⊂U.

Corollary 3.31. If f:X→Y is almost δgb-closed,then each δ-closed set K of Y and

Proof: Suppose that f:X→Y is almost δgb-closed.Let K be any δ-closed set of Y and U be any regular-closed set containing

f^{-1}(K). By Lemma 3.30, there exists a δgb-open set H of Y such that K⊂H and

f^{-1}(H)⊂U.Since K is δ-closed, then by Theorem 3.15, K⊂bint(H).Put bint(H)=V, then V is a b-open set such that K⊂V and f^{-1}(V)⊂U.

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