ORTHOGONAL REVERSE SEMIDERIVATIONS ON SEMIPRIME SEMIRING

U. REVATHY, R.MURUGESAN, S.SOMASUNDARAM

Abstract: Motivated by some results on Semi prime Gamma Rings with Orthogonal Reverse Derivations, published in the International Journal of Pure and Applied mathematics vol. 83(2013) pages(233-245) by Kalyan Kumar Dey, Akhil Chandra Paul, Isamiddin S.Rakhimov, the authors defined the notion of Reverse Derivations on Gamma Rings and investigated some results. In this paper, we introduce the notion of Orthogonal Reverse Semi derivations of Semi prime Semi rings and derived some interesting results.

Keywords: Semi rings, Reverse Derivations, Reverse Semi derivation, Orthogonal Reverse Semi derivation.

Introduction: This paper has been inspired by the work of Kalyan Kumar Dey, Akhil Chandra Paul, Isamiddin S.Rakhimov [4]. Bresar and Vukman [1] initiated the notion of orthogonality for two derivations on a semiprime ring, and they obtained several necessary and sufficient conditions for two derivations to be orthogonal. They also obtained a counter part of a result of Posner from [6]. In [5] the authors introduced the Some results on semi derivations on semi prime semi ring. In this paper, we introduce the notion of orthogonality of two reverse semi derivations on semi prime semi ring and we presented several necessary and sufficient conditions for two reverse semi derivations to be orthogonal.

2.Preliminaries

Definition: 2.1

A semiring (S,+,•) is a non-empty set S together with two binary operations, + and • such that

1. (S,+) and (S,•) are a Semigroup.
2. For all a, b, c ϵ S, a. (b + c) = a. b + a. c and (b + c). a = b. a + c. a

Definition: 2.2: A semiring S is said to be 2-torsionfree if 2x = 0 ⇒ x = 0, ∀x ∈ S.

Definition: 2.3

A semiring S is prime if xSy = 0 ⇒ x = 0 or y = 0, ∀ x,y ∈ S and S is semiprime if xSx = 0⇒ x = 0,∀ x ∈ S.

Definition: 2.4

An additive map d: S → S is called a derivation if d(xy)=d(x) y + x d(y),∀ x, y ∈ S

Definition: 2.5

Let d, g be two additive maps from S to S. They are said to be orthogonal if d(x)Sg(y) = 0 = g(y)Sd(x), ∀ x, y ∈ S.

We write [x, y] = x y – y x and note that important identity [xy, z] = x [y, z] + [x, z] y and [x, yz] = y [x, z] + [x, y] z

3. Reverse Semiderivations and Orthogonal Reverse Semiderivations

Definition:3.1

In a semiring S, if d is an additive mapping from S into itself satisfying d(xy) = d(y) x + y d(x), ∀ x, y ∈ S, then d is called a reverse derivation on S.

Example:3.2

Let S $=\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} / a,b \in Z^+ \cup \{0\}\right\}$ be a semiring. Then d : S → S defined by $d\left[\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right] = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is a reverse derivation.

Definition:3.3

An additive mapping $d : S \to S$ is called semiderivation associated with a function $d_1 : S \to S$, ∀ x, y ∈ S

(i) d(xy)=d(x) d_1 (y)+x d(y)=d(x)y+ d_1 (x)d(y)

(ii) d(d_1 (x)) = d_1 (d(x))

If d_1 = I, that is, an identity mapping of S then all semiderivations associated with d_1 are merely ordinary derivations. If d_1 is any endomorphism of S, then semiderivations are of the form d(x) = x - d_1 (x)

Definition:3.4: An additive mapping d : S → S is called semiderivation associated with a function d_1 :S → S satisfying d(xy) = d(y) d_1 (x) + y d(x), ∀ x, y ∈ S is called a reverse semiderivation.

Note:

- If S is Commutative, then both semiderivation and reverse semiderivation are the same.
- In general reverse semiderivation is not a semiderivation but it is a Jordan semiderivation.
- A non-zero reverse semiderivation cannot be orthogonal on itself.

Lemma:3.5: Let S be a 2-torsion free semiprime semiring and a,b ∈ S. Then the following conditions are equivalent

(i) axb = 0, $\forall x \in S$

(ii) bxa = 0, $\forall x \in S$

(iii) axb + bxa = 0, $\forall x \in S$. If one of the conditions is fulfilled, then ab = ba = 0.

Lemma:3.6

Let S be a semiprime semiring and suppose that additive mappings d and g on S into itself satisfy d(x) S g(x) = 0, $\forall x \in S$. Then d(x) S g(y) = 0, $\forall x, y \in S$.

Theorem:3.7

Let S be a 2-torsion free semiprime semiring. Let d and g be reverse semi derivations on S. Then d(x)g(y)+g(x) d(y)= 0, $\forall x, y \in S$ iff d and g are orthogonal.

Proof:

Suppose that d(x)g(y)+g(x) d(y)=0, $\forall x, y \in S$

Put y = xy

d(x) g(xy) + g(x) d(xy) = 0, $\forall x, y \in S$

⇒ d(x)[g(y) g_1 (x) + y g(x)] + g(x) [d(y) d_1 (x)+ y d(x)] = 0, $\forall x, y \in S$

⇒d(x) g(y) g_1 (x) + d(x) y g(x) + g(x) d(y) d_1 (x)+ g(x) y d(x) = 0, $\forall x, y \in S$

Since d_1 and g_1 are surjective we have,

d(x) g(y) x + d(x) y g(x) + g(x) d(y) x +g(x) y d(x) = 0, $\forall x, y \in S$

⇒ [d(x) g(y) + g(x) d(y)] x + d(x) y g(x) +g(x) y d(x) = 0, $\forall x, y \in S$

⇒ d(x) y g(x) + g(x) y d(x) = 0, $\forall x, y \in S$

By lemma 3.5, d(x) y g(x) = 0, $\forall x, y \in S$

By lemma 3.6, d(x) y g(z) = 0, $\forall x, y \in S$

By lemma 3.5, g(z) y d(x) = 0, $\forall x, y \in S$

Thus d(x) y g(z) = 0 = g(z) y d(x), $\forall x, y \in S$

∴ d(x) S g(z) = 0 = g(z) S d(x), $\forall x, y \in S$

∴ d and g are Orthogonal.

Conversely, if d and g are Orthogonal.

∴ d(x)S g(y) = 0 = g(y) S d(x), $\forall x, y \in S$

ie, d(x) s g(y) = 0 = g(y) s d(x), $\forall x, y \in S$

By lemma 3.5, d(x)g(y)=0=g(x)d(y), $\forall x, y \in S$

∴ d(x) g(y) + g(x) d(y) = 0, $\forall x, y \in S$

Remark:

Let d and g be reverse semiderivations on a semiring. Then the following results are hold good

1. dg(xy) = d(x) g(y) + x dg(y) + dg(x) y + g(x) d(y)
2. gd(xy) = g(x) d(y) + x gd(y) + gd(x) y + d(x) g(y)

Theorem:3.8

Let S be a 2-torsion free semiprime semiring and d and g be reverse semiderivations on S. Then d and g are orthogonal iff d(x) g(x) = 0, $\forall x \in S$

Proof:

(i)⟺ (ii) Assume that d and g are orthogonal

∴ d(x) s g(y) = 0 = g(y) s d(x), $\forall x, y, s \in S$

Now consider d(x) s g(y)=0, $\forall x, y, s \in S$. Using lemma 3.6, we get d(x) s g(x)=0

Using lemma 3.5, d(x) g(x)=0, $\forall x \in S$

Conversely, assume that d(x)g(x)= 0, $\forall x \in S$.

The linearization of d(x + y) g(x + y) = 0 gives

d(x) g(y) + d(y) g(x) = 0, $\forall x, y \in S$ (1)

Take y = yz in (1), d(x) g(yz) + d(yz) g(x) = 0, $\forall x, y, s \in S$

⇒d(x) [g(z) g_1 (y) + z g(y)] + [d(z) d_1 (y) + z d(y)] g(x) = 0, $\forall x, y, s \in S$

⇒d(x)[g(z)y+zg(y)]+[d(z)y+zd(y)]g(x)=0, $\forall x, y, s \in S$ [∵ d_1 and g_1 are surjective]

⇒ d(x) g(z) y + d(x) z g(y) + d(z) y g(x) + z d(y) g(x) = 0, $\forall x, y, s \in S$ (2)

Since d(x) g(z) = $-d(z)g(x)$ and

d(y) g(x) = $-d(x)g(y)$

(2)⇒

$$-d(z)g(x)+d(x)zg(y)+d(z)yg(x)-zd(x)g(y)=0$$

$$\Rightarrow -d(z)[yg(x)-g(x)y+d(x)z-zd(x)]g(y)=0$$

$$\Rightarrow d(z)[y,g(x)]+[d(x),z]g(y)=0$$

Replacing z by d(x).

d(d(x))[y,g(x)]+[d(x),d(x)]g(y)=0

⇒ d^2(x)[y,g(x)] = 0 (3)

Let y = yw, $\forall y, w \in S$ in the above equation d^2(x) [yw, g(x)] = 0

⇒ d^2(x) y[w,g(x)] + d^2(x) [y,g(x)] w = 0

⇒ d^2(x) y[w,g(x)] =0, $\forall x, y, w \in S$ [∵ by (3)]

Using lemma 3.6, d^2(x)y[w, g(y)]=0, $\forall x, y, w \in S$ (4)

Replacing x by xu and using the remark,

0 = d^2(xu) y [w, g(y)] = 0, $\forall x, y, u, w \in S$

= [d^2(x) u+2d(x)d(u)+ x d^2(u)]y [w,g(y)]

= 2 d(x)d(u) y [w, g(y)] [∵ by (4)]

Since S is 2-torsionfree, d(x)d(u)y[w,g(y)]= 0, $\forall x, y, s \in S$ (5)

Take x = xz, d(xz) d(u) y [w, g(y)] = 0

d(z) d_1 (x)d(u)y[w,g(y)]+zd(x)d(u)y[w,g(y)]=0 ⇒ d(z) x d(u) y [w,g(y)] = 0

Inparticular, d(z) x d(x) y [w, g(y)]=0

Take d(z) = d(x) y [w,g(y)], then

d(x)y[w,g(y)] x d(x) y [w,g(y)] = 0

Since S is semiprime , d(x) y [w,g(y)] = 0

d(x) g(y) = g(y) d(x), $\forall x, y \in S$.

∴ (1) ⇒ g(y) d(x) + d(y) g(x) = 0

Using theorem 3.7 we get d and g are orthogonal.

Theorem:3.9

Let S be a 2-torsion free semiprime semiring and d and g be reverse semiderivations on S. Then the following conditions are equivalent

1. d and g are orthogonal
2. d(x) g(x) = 0, ∀ x ∈ S
3. g(x) d(x) = 0, ∀ x ∈ S
4. d(x) g(x) + g(x) d(x) = 0, ∀ x ∈ S

Proof:

(i)⟺ (ii) The proof is immediate from the previous theorem 3.8

The proof of (i) ⟺(iii)is the similar proof to that of (i) ⟺(ii) (i)⟺ (iv) The proof is immediate from the theorem 3.7

Theorem:3.10

Let S be a 2 – torsion free semiprime semiring. Let d and g be reverse semiderivations on S. Then the following conditions are equivalent

(i) d and g are Orthogonal
(ii) dg = 0
(iii) gd = 0
(iv) dg + gd = 0
(v) dg is a derivation
(vi) gd is a derivation

Proof:

(i)⟺ (ii) Suppose d and g are Orthogonal

d(x)Sg(y)=0=g(y)Sd(x), $\forall x, y \in S$ ⇒d(x)sg(y)= 0 = g(y) s d(x), $\forall x, y, s \in S$

Now , d(x) s g(y) = 0⇒d[d(x) s g(y)] = d(0)

i.e, d[g(y)]sd(x)+g(y)d(s)d(x)+sg(y)d[d(x)]= 0

Since d and g are orthogonal, dg(y)sd(x) = 0.

Replacing x by g(y), dg(y) s dg(y) = 0

Since S is semiprime, dg(y) =0, $\forall y \in S$

∴ dg = 0

Conversely, Suppose dg = 0

Now by remark dg (xy) = d (x) g (y) + x dg (y) + dg(x) y + g (x) d (y), $\forall x, y \in S$ (1)

Since dg=0,(1)⇒0=d(x)g(y)+g(x)d(y), $\forall x, y \in S$

By lemma 3.7, d and g are orthogonal

The Proof of (i)⟺ (iii) is the similar proof to that of (i)⟺ (ii)

(i) ⟺ (iv) Suppose d and g are orthogonal

Hence dg = 0 and gd = 0

∴ dg + gd = 0

Conversely, Suppose that dg + gd = 0 (2)

Then, (dg + gd) (xy) = dg (xy) + gd (xy) = 0

By the remark,

d(x)g(y)+xdg(y)+ dg(x)y + g(x)d(y)+g(x)d(y)+ x gd(y) + gd(x) y + d (x) g(y)=0

2[d(x) g(y) + g(x) d(y)] + (dg + gd)(x) y + x (dg+gd) (y) = 0

⇒2 [d(x) g(y) + g(x) d(y)] = 0 [∵ by (2)]

⇒d (x) g (y) + g(x) d(y) = 0 [∵ S is 2-torsion free]

⇒By lemma 3.7, d and g are orthogonal

(i)⟺ (v) Suppose dg is derivation

dg (xy) = dg(x) y + x dg (y), $\forall x, y \in S$ (3)

By remark, dg (xy) = d(x) g(y) + x dg (y) + dg (x) y + g (x) d (y), $\forall x, y \in S$ (4)

Using (3) & (4), dg (xy) = d(x) g(y) + dg(xy) +g (x) d (y), $\forall x, y \in S$

0 = d(x) g(y) + g(x) d(y), $\forall x, y \in S$ [∵ S is additively cancellative]

∴ d and g are orthogonal.

Conversely, suppose d and g are orthogonal

By lemma 3.7, d(x) g(y) +g (x) d (y) =0

Using this is (4), dg (xy) = dg (x) y + x dg (y)

∴ dg is a derivation

The proof of (i)⟺ (vi) is the similar proof to that of (i) ⟺ (v)

Corollary:3.11

Let S be a 2- torsion free semiring. Suppose that d and g are orthogonal reverse semiderivations on S. Then either d=0 or g=0

The proof is immediate from the previous theorem.

Theorem:3.12

Let S be a 2- torsion free semiprime semiring and let d and g be reverse semiderivations on S. Suppose $d^2 = g^2$, then $d + g \, and \, d - g$ are orthogonal

Proof:

Let d and g be reverse semiderivations on S. Suppose $d^2 = g^2$ (1)

For all x∈ S, [(d+g) (d−g) + (d−g) (d+g)] x

=(d+g)(d−g)(x) + (d−g) (d+g) (x)

=(d+g)(d(x)−g(x))+(d−g)(d(x)+g(x))

= d(d(x)−dg(x)+gd(x)−gg(x)

+ d(d(x) + dg (x) – gd(x) – g(g(x))

$= d^2(x) - dg(x) + gd(x) - d^2(x) + g^2(x) + dg(x) - gd(x) - g^2(x)$

= 0

By the previous theorem, (d+g) and (d−g) are orthogonal.

Theorem:3.13

Let S be a 2- torsion free semiprime semiring and let d and g be reverse semiderivations of S.

If d(x) d(x) = g(x)g(x), then d + g and d−g are orthogonal.

The proof is similar to the previous theorem.

References:

1. Bresar.M and Vukman.J, Orthogonal derivation and extension of a theorem of Posner, Radovi Matematicki 5(1989), 237-246.
2. *Dr. C. Jaya Subba Reddy D. Prabhakara Reddy P. Gurivi Reddy,* Rings With Associators in the Right Nucleus; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 1 (2013), Pg 57-58
3. Chandramouleeswaran, Revathy, Orthogonal Derivations on Semirings, M.Phil. Dissertation submitted to Madurai Kamaraj University (2012), 20-47 [4] Jonathan S.Golan, Semirings and their Applications, Kluwer Academic Press(1969).
4. *Anindita Basu,* Fuzzy Composite Modular Sequence Space; Mathematical Sciences

International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 229-235
5. Kazimierz Glazek, A Guide to the literature on Semirings and their applications in Mathematics.
6. Mohd Nadeem, Jaydev Dabas, Existence and Uniqueness of Solution for Impulsive; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 61-68
7. Kalyan Kumar Dey, Akhil Chandra Paul, Isamiddin S.Rakhimov, Semiprime Gamma Rings with Orthogonal Reverse Derivations, International Journal of Pure and Applied mathematics, Vol 83 (2013), 233-245
8. D.V.S.R. Anil Kumar, Y.V. Seshagiri Rao, on Various Decompositions of A Signed Lattice Measure; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 604-606
9. K.Kanak Sindhu, R.Murugesan, P.Namasivayam, Some results on semiderivations of semiprime semirings, International journal of scientific and research publications, Vol 5, 2550-3153
10. E.Posner, Derivations in primerings, Proc.Amer Math.Soc.8(1957), 1093-1100
11. *M. Lellis Thivagar, B. Meeradevi, G. Jaya Parthasarathy,* New Type of Supra Generalized Closed Sets; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Spl Issue (2014), Pg 916-923

***

U.Revathy/ Assistant Professor / Thiruvalluvar College/Papanasam-627425/Manonmaniam Sundaranar university/ Tamil Nadu/India.
R.Murugesan/Associate Professor & Head/Thiruvalluvar College/Papanasam-627425/Manonmaniam Sundaranar University/Tamil Nadu/India,
S.Somasundaram/Professor/Manonmaniam SundaranarUniversity/Tirunelveli-627012/ TamilNadu/ India.

.