FS-SETS AND INFINITE DISTRIBUTIVE LAWS

VADDIPARTHI YOGESWARA, BISWAJIT RATH, CH.RAMASANYASI RAO

Abstract: In this paper we introduce the concept of Infinite Distributive Lawsfor any given family of Fs-subset of a given Fs-set.

Keywords: Fs-set, Fs-subsets, Family of Fs-subset, Fs-Infinite Distributive Laws

Introduction: Ever since Zadeh [18] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets.

Murthy[8] introduced f-set in order to prove Axiom of choice for fuzzy sets which is not true for L-fuzzy sets introduced by Goguen[11]. Murthy [10] introduced the definition of f-complement of an f-subset in [10]. We can easily see that the collection all f-subsets of a given f-set with this definition of f-complement could not form a Boolean algebra. Recently many researchers put their efforts in order to prove collection of all fuzzy subsets of given fuzzy set is Boolean algebra under suitable operations and it seems among them the efforts of Tridiv [4],[5],[20] are most successful. The definition of fuzzy set given by Tridiv is based on the definition of fuzzy set given by H.K. Baruah [6]. Particularly in the definition of membership function of Tridiv [4] namely, $\mu_1(x) - \mu_2(x)$, $-\mu_2(x)$ will not be in the real interval [0,1].To eliminate this lacuna Vaddiparthi Yogeswara , G.Srinivas and Biswajit Rath introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions .They are successful in their efforts in proving that result with some conditions. In the paper [1], distributive laws for finite Fs-subsets of a given Fs-set are proved with some conditions. In this paper we introduce the concept of Infinite Distributive Lawsfor a given family of Fs-subsets of a given Fs-set..Here the operations on collection of Fs-subsets of $\mathcal{A}$ are Fs-union, Fs-intersection and Fs-complement. For smooth reading of the paper, the theory of Fs-sets in brief is dealt with in first two sections. We denote the largest element of a complete Boolean algebra L_A[1.1] by M_A or 1_A. We denote Fs-union and crisp set union by the same symbol ∪ and similarly Fs-intersection and crisp set intersection by the same symbol ∩. For all lattice theoretic properties and Boolean algebraic properties one can refer Szasz [13], Garret Birkhoff[14],Steven Givant • Paul Halmos[12] and Thomas Jech[15].

Theory Of Fs-Sets:

1.1 Fs-set: Let U be a universal set, $A_1 \subseteq U$ and let $A \subseteq U$ be non-empty. A four tuple $\mathcal{A} = (A_1, A, \overline{A}(\mu_{1A_1}, \mu_{2A}), L_A)$ is said be an Fs-set if, and only if

(1) $A \subseteq A_1$

(2) L_A is a complete Boolean Algebra

(3) $\mu_{1A_1}: A_1 \longrightarrow L_A$,$\mu_{2A}: A \longrightarrow L_A$,are functions such that $\mu_{1A_1}|A \geq \mu_{2A}$

(4) $\overline{A}$:A⟶ L_A is defined by $\overline{A}x = \mu_{1A_1}x \wedge (\mu_{2A}x)^c$, for each $x \in A$

1.2 Fs-subset

Let $\mathcal{A}=\left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$ and $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$ be a pair of Fs-sets. $\mathcal{B}$ is said to be an Fs-subset of $\mathcal{A}$, denoted by $\mathcal{B} \subseteq \mathcal{A}$, if, and only if

(1) $B_1 \subseteq A_1$, $A \subseteq B$

(2) L_B is a complete subalgebra of L_A or $L_B \leq L_A$

(3) $\mu_{1B_1} \leq \mu_{1A_1}|B_1$, and $\mu_{2B}|A \geq \mu_{2A}$

1.3 Proposition: Let $\mathcal{B}$ and $\mathcal{A}$ be a pair of Fs-sets such that $\mathcal{B} \subseteq \mathcal{A}$. Then $\overline{B}x \leq \overline{A}x$ is true for each $x \in A$

1.4 Definition: For some L_X, such that $L_X \leq L_A$ a four tuple $\mathcal{X} = \left(X_1, X, \overline{X}\left(\mu_{1X_1}, \mu_{2X}\right), L_X\right)$ is not an Fs-set if, and only if

(a) $X \nsubseteq X_1$ or

(b) $\mu_{1X_1}x \ngeq \mu_{2X}x$, for some $x \in X \cap X_1$

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of $\mathcal{B}$ for any $\mathcal{B} \subseteq \mathcal{A}$.

Definition: An Fs-subset $\mathcal{Y}=\left(Y_1, Y, \overline{Y}\left(\mu_{1Y_1}, \mu_{2Y}\right), L_Y\right)$ of $\mathcal{A}$, is said to be an Fs-empty set of second kind if, and only if

(a') $Y_1 = Y = A$

(b') $L_Y \leq L_A$

(c') $\overline{Y} = 0$

1.4.1 Remark: we denote Fs-empty set of first kind or Fs-empty set of second kind by $\Phi_{\mathcal{A}}$and we prove later (1.15), $\Phi_{\mathcal{A}}$ is the least Fs-subset among all Fs-subsets of $\mathcal{A}$.

1.5Definition: et$\mathcal{B}_1 = \left(B_{11}, B_1, \overline{B}_1\left(\mu_{1B_{11}}, \mu_{2B_1}\right), L_{B_1}\right)$ and $\mathcal{B}_2 = \left(B_{12}, B_2, \overline{B}_2\left(\mu_{1B_{12}}, \mu_{2B_2}\right), L_{B_2}\right)$ be a pair of Fs-subsets.

(1) We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are (1,5)-equal, if $B_{11} = B_{12}$ and $L_{B_1} = L_{B_2}$
(2) We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are (2,5)-equal, if $B_1 = B_2$ and $L_{B_1} = L_{B_2}$
(3) We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are 3-equal, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are (1,5)-equal and $\mu_{1B_{11}} = \mu_{1B_{12}}$
(4) We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are 4-equal, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are (2,5)-equal and $\mu_{2B_1} = \mu_{2B_2}$
(5) We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Total equal denoted $\mathcal{B}_1 = \mathcal{B}_2(T)$, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are (2,5)-equal and $\overline{B}_1 = \overline{B}_2$
(6) We say that $\mathcal{B}_1$, $\mathcal{B}_2$ are Full-equal, denoted $\mathcal{B}_1 = \mathcal{B}_2$, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are 3-equal and 4-equal.

1.6 Proposition:
$\mathcal{B}_1 = \left(B_{11}, B_1, \overline{B}_1\left(\mu_{1B_{11}}, \mu_{B_1}\right), L_{B_1}\right)$ and $\mathcal{B}_2 = \left(B_{12}, B_2, \overline{B}_2\left(\mu_{1B_{12}}, \mu_{B_2}\right), L_{B_2}\right)$ are equal if, only if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$

1.7 Definition of Fs-union for a given pair of Fs-subsets of $\mathcal{A}$:
Let $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$ and $\mathcal{C}=\left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$, be a pair of Fs-subsets of $\mathcal{A}$. Then,
the Fs-union of $\mathcal{B}$ and $\mathcal{C}$, denoted by $\mathcal{B}\cup\mathcal{C}$ is defined as
$\mathcal{B}\cup\mathcal{C}=\mathcal{D}=\left(D_1, D, \overline{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$, where
(1) $D_1 = B_1 \cup C_1$, $D = B \cap C$
(2) $L_D = L_B \vee L_C$=complete subalgebra generated by $L_B \cup L_C$
(3) $\mu_{1D_1}: D_1 \longrightarrow L_D$ is defined by

$$\mu_{1D_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$$

$\mu_{2D}: D \longrightarrow L_D$ is defined by

$$\mu_{2D}x = \mu_{2B}x \wedge \mu_{2C}x$$

$\overline{D}: D \longrightarrow L_D$ is defined by

$$\overline{D}x = \mu_{1D_1}x \wedge \left(\mu_{2D}x\right)^c$$

1.8 Proposition: $\mathcal{B}\cup\mathcal{C}$ is an Fs-subset of $\mathcal{A}$.

1.9 Definition of Fs-intersection for a given pair of Fs-subsets of $\mathcal{A}$:
Let $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$ and $\mathcal{C}=\left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ be a pair of Fs-subsets of $\mathcal{A}$ satisfying the following conditions:
(i) $B_1 \cap C_1 \supseteq B \cup C$
(ii) $\mu_{1B_1}x \wedge \mu_{1C_1}x \geq (\mu_{2B} \vee \mu_{2C})x$, for each $x \in A$
Then, the Fs-intersection of $\mathcal{B}$ and $\mathcal{C}$, denoted by $\mathcal{B}\cap\mathcal{C}$ is defined as
$\mathcal{B}\cap\mathcal{C}=\mathcal{E}=\left(E_1, E, \overline{E}\left(\mu_{1E_1}, \mu_{2E}\right), L_E\right)$, where
(a) $E_1 = B_1 \cap C_1$, $E = B \cup C$
(b) $L_E = L_B \wedge L_C = L_B \cap L_C$
(c) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is defined by $\mu_{1E_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$
$\mu_{2E}: E \longrightarrow L_E$ is defined by

$$\mu_{2E}x = (\mu_{2B} \vee \mu_{2C})x$$

$\overline{E}: E \longrightarrow L_E$ is defined by
$\overline{E}x = \mu_{1E_1}x \wedge \left(\mu_{2E}x\right)^c$.

1.9.1 Remark: If (i) or (ii) fails we define $\mathcal{B}\cap\mathcal{C}$ as $\mathcal{B}\cap\mathcal{C}=\Phi_{\mathcal{A}}$, which is the Fs-empty set of first kind.

1.10 Proposition: For any pair of Fs-subsets $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$ and $\mathcal{C}=\left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ of $\mathcal{A}$, the following results are true
(1) $\mathcal{B}\subseteq\mathcal{B}\cup\mathcal{C}$ and $\mathcal{C}\subseteq\mathcal{B}\cup\mathcal{C}$
(2) $\mathcal{B}\cap\mathcal{C}\subseteq\mathcal{B}$ and $\mathcal{B}\cap\mathcal{C}\subseteq\mathcal{C}$ provided $\mathcal{B}\cap\mathcal{C}$ exists
(3) $\mathcal{B}\subseteq\mathcal{C}$ implies $\mathcal{B}\cup\mathcal{C}=\mathcal{C}$
(4) $\mathcal{B}\cap\mathcal{C}=\mathcal{B}$ when $\mathcal{B}\neq \Phi_{\mathcal{A}}$ and $\mathcal{B}\subseteq\mathcal{C}$ and $\Phi_{\mathcal{A}} \cap\mathcal{C}=\Phi_{\mathcal{A}}$
(5) $\mathcal{B}\cup\mathcal{C}= \mathcal{C} \cup \mathcal{B}$ (commutative law of Fs-union)
(6) $\mathcal{B}\cap\mathcal{C}= \mathcal{C} \cap \mathcal{B}$ provided $\mathcal{B}\cap\mathcal{C}$ exists. (commutative law of Fs-intersection)
(7) $\mathcal{B}\cup\mathcal{B}=\mathcal{B}$
(8) $\mathcal{B}\cap\mathcal{B}=\mathcal{B}$ ((7)and (8) are Idempotent laws of Fs-union and Fs-intersection respectively)

1.11 Proposition: For any Fs-subsets $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{A} = \left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$,
the following associative laws are true:
(I) $\mathcal{B} \cup (\mathcal{C} \cup \mathcal{D}) = (\mathcal{B} \cup \mathcal{C}) \cup \mathcal{D}$
(II) $\mathcal{B} \cap (\mathcal{C} \cap \mathcal{D}) = (\mathcal{B} \cap \mathcal{C}) \cap \mathcal{D}$, whenever Fs-intersections exist.

1.12 Arbitrary Fs-unions and arbitrary Fs-intersections:
Given a family $(\mathcal{B}_i)_{i\in I}$ of Fs-subsets of $\mathcal{A}=\left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$, where
$\mathcal{B}_i = \left(B_{1i}, B_i, \overline{B}_i\left(\mu_{1B_{1i}}, \mu_{2B_i}\right), L_{B_i}\right)$, for any $i \in I$

1.13 Definition of Fs-union is as follows
Case (1): For $I=\Phi$, define Fs-union of $(\mathcal{B}_i)_{i\in I}$, denoted by $\cup_{i\in I}\mathcal{B}_i$ as $\cup_{i\in I}\mathcal{B}_i = \Phi_{\mathcal{A}}$, which is the Fs-empty set
Case (2): Define for $I\neq\Phi$, Fs-union of $(\mathcal{B}_i)_{i\in I}$ denoted by $\cup_{i\in I}\mathcal{B}_i$ as follow

$$\bigcup_{i\in I}\mathcal{B}_i = \mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right),$$

where
(a) $B_1 = \cup_{i\in I} B_{1i}$, $B = \cap_{i\in I} B_i$
(b) $L_B = \vee_{i\in I} L_{B_i}$ = complete subalgebra generated by $\cup L_i (L_i = L_{B_i})$
(c) $\mu_{1B_1}: B_1 \rightarrow L_B$ is defined by
$\mu_{1B_1}x = \left(\vee_{i\in I}\mu_{1B_{1i}}\right)x = \vee_{i\in I_x}\mu_{1B_{1i}}x$, where

$$I_x = \{i \in I | \ x \in B_i\}$$

$\mu_{2B}: B \longrightarrow L_B$ is defined by $\mu_{2B}x = \left(\wedge_{i\in I}\mu_{2B_i}\right)x$
$=\wedge_{i\in I}\mu_{2B_i}x$

$\overline{B}: B \longrightarrow L_B$ is defined by $\overline{B}x = \mu_{1B_1}x \wedge (\mu_{2B}\, x)^c$

1.13.1Remark: We can easily show that (d) $B_1 \supseteq B$ and $\mu_{1B_1}|B \geq \mu_{2B}$.

1.14 Definition of Fs-intersection:

Case (1): For I=Φ, we define Fs-intersection of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcap_{i\in I}\mathcal{B}_i$ as $\bigcap_{i\in I}\mathcal{B}_i = \mathcal{A}$

Case (2): Suppose $\bigcap_{i\in I} B_{1i} \supseteq \bigcup_{i\in I} B_i$ and $\bigwedge_{i\in I}\mu_{1B_{1i}}|(\bigcup_{i\in I} B_i) \geq \bigvee_{i\in I}\mu_{2B_i}$

Then, we define Fs-intersection of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcap_{i\in I}\mathcal{B}_i$ as follows

$$\bigcap_{i\in I}\mathcal{B}_i = \mathcal{C} = \left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$$

(a') $C_1 = \bigcap_{i\in I} B_{1i}\, , C = \bigcup_{i\in I} B_i$

(b') $L_C = \bigwedge_{i\in I} L_{B_i}$

(c') $\mu_{1C_1}: C_1 \longrightarrow L_C$ is defined by $\mu_{1C_1}x = \left(\bigwedge_{i\in I}\mu_{1B_{1i}}\right)x = \bigwedge_{i\in I}\mu_{1B_{1i}}\, x$

$\mu_{2C}: C \longrightarrow L_C$ is defined by

$\mu_{2C}x = \left(\bigvee_{i\in I}\mu_{2B_i}\right)x = \bigvee_{i\in I_x}\mu_{2B_i}x,$ where, $I_x = \{i \in I|\, x \in B_i\}$

$\overline{C}: C \longrightarrow L_C$ is defined by $\overline{C}x = \mu_{1C_1}x \wedge (\mu_{2C}\, x)^c$

Case (3): $\bigcap_{i\in I} B_{1i} \not\supseteq \bigcup_{i\in I} B_i$ or $\bigwedge_{i\in I}\mu_{1B_{1i}}|(\bigcup_{i\in I} B_i) \not\geq \bigvee_{i\in I}\mu_{2B_i}$

We define $\bigcap_{i\in I}\mathcal{B}_i = \Phi_{\mathcal{A}}$

1.14.1Lemma: For any Fs-subset $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_A\right)$ and $\mathcal{B} \subseteq \mathcal{B}_i = \left(B_{1i}, B_i, \overline{B}_i\left(\mu_{1B_{1i}}, \mu_{2B_i}\right), L_{B_i}\right)$

1.15 Proposition:$(\mathcal{L}(\mathcal{A}), \cap)$ is $\wedge$-complete lattics.

1.15.1 Corollary: For any Fs-subset $\mathcal{B}$ of$\mathcal{A}$, the following results are true

(i) $\Phi_{\mathcal{A}} \cup \mathcal{B} = \mathcal{B}$

(ii) $\Phi_{\mathcal{A}} \cap \mathcal{B} = \Phi_{\mathcal{A}}$.

1.16 Proposition:$(\mathcal{L}(\mathcal{A}), \cup)$ is $\vee$-complete lattics.

1.16.1 Corollary:$(\mathcal{L}(\mathcal{A}), \cup, \cap)$ is a complete lattice with$\vee$and$\wedge$

1.17Proposition: Let$\mathcal{B}\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, $\mathcal{C}=\left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$and $\mathcal{D}=\left(D_1, D, \overline{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$.Then $\mathcal{B}\cup(\mathcal{C}\cap\mathcal{D})=(\mathcal{B}\cup\mathcal{C})\cap(\mathcal{B}\cup\mathcal{D})$ provided $\mathcal{C}\cap\mathcal{D}$ exists.

1.18Proposition: Let $\mathcal{B}\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, $\mathcal{C}=\left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$and $\mathcal{D}=\left(D_1, D, \overline{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$.Then $\mathcal{B}\cap(\mathcal{C}\cup\mathcal{D})=(\mathcal{B}\cap\mathcal{C})\cup(\mathcal{B}\cap\mathcal{D})$ provided in R.H.S$(\mathcal{B}\cap\mathcal{C})$ and $(\mathcal{B}\cap\mathcal{D})$ exist.

2. Fs-Distributive Laws for the Family of Fs-subsets of Fs-set

2.1Proposition: Let $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$ and $\mathcal{C}_i = \left(C_{1i}, C_i, \overline{C}_i\left(\mu_{1C_{1i}}, \mu_{2C_i}\right), L_{C_i}\right)$ be Fs-subsets of $\mathcal{A}$ for i ∈ I, then the following statements are true

(A) $\mathcal{B} \cup (\bigcap_{i\in I}\mathcal{C}_i) = \bigcap_{i\in I}(\mathcal{B}\cup\mathcal{C}_i)$, provided $\bigcap_{i\in I}\mathcal{C}_i$ exists for i ∈ I

(B) $\mathcal{B} \cap (\bigcup_{i\in I}\mathcal{C}_i) = \bigcup_{i\in I}(\mathcal{B}\cap\mathcal{C}_i)$, provided $\mathcal{B}\cap\mathcal{C}_i$ exists for i ∈ I

Proof (A)

Case (I) For $I = \Phi, \bigcap_{i\in I}\mathcal{C}_i = \mathcal{A}$

LHS: $\mathcal{B} \cup (\bigcap_{i\in I}\mathcal{C}_i) = \mathcal{A}$, RHS: $\bigcap_{i\in I}(\mathcal{B}\cup\mathcal{C}_i) = \mathcal{A}$

Hence LHS = RHS

Case (II) For $I \neq \Phi, \bigcap_{i\in I}\mathcal{C}_i = \mathcal{D} = \left(D_1, D, \overline{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$, where

(1) $D_1 = \bigcap_{i\in I} C_{1i}\, ,\ D = \bigcup_{i\in I} C_i$

(2) $L_D = \bigwedge_{i\in I} L_{C_i}$

(3) $\mu_{1D_1}: D_1 \to L_D$ is defined by $\mu_{1D_1}x = \left(\bigwedge_{i\in I}\mu_{1C_{1i}}\right)x = \bigwedge_{i\in I}\mu_{1C_{1i}}\, x$

$\mu_{2D}: D \to L_D$ is defined by $\mu_{2D}x = \left(\bigvee_{i\in I}\mu_{2C_i}\right)x$

$\overline{D}: D \to L_D$ is defined by $\overline{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c$

$\bigcap_{i\in I}\mathcal{C}_i = \mathcal{D}$ exists for i ∈ I implies

(I) $D_1 = \bigcap_{i\in I} C_{1i} \supseteq \bigcup_{i\in I} C_i = D$

(II) $\mu_{1D_1}x = \left(\bigwedge_{i\in I}\mu_{1C_{1i}}\right)x \geq \left(\bigvee_{i\in I}\mu_{2C_i}\right)x = \mu_{2D}x$

$$\mathcal{B} \cup \left(\bigcap_{i\in I}\mathcal{C}_i\right) = \mathcal{B}\cup\mathcal{D} = \mathcal{E} = \left(E_1, E, \overline{E}\left(\mu_{1E_1}, \mu_{2E}\right), L_E\right)$$

(4) $E_1 = B_1 \cup D_1 = B_1 \cup (\bigcap_{i\in I} C_{1i}) = \bigcap_{i\in I}(B_1 \cup C_{1i})$, $E = B \cap D = B \cap (\bigcup_{i\in I} C_i) = \bigcup_{i\in I}(B \cap C_i)$

(5) $L_E = L_B \vee L_D = L_B \vee \left(\bigwedge_{i\in I} L_{C_i}\right)$

(6) $\mu_{1E_1}: E_1 \to L_E$ is given by $\mu_{1E_1}x = \left(\mu_{1B_1} \vee \mu_{1D_1}\right)x = \left[\mu_{1B_1} \vee \left(\bigwedge_{i\in I}\mu_{1C_{1i}}\right)\right]x$

$\mu_{2E}: E \to L_E$ is given by $\mu_{2E}x = (\mu_{2B} \wedge \mu_{2D})x = \left[\mu_{2B} \wedge \left(\bigvee_{i\in I}\mu_{2C_i}\right)\right]x$

$\overline{E}: E \to L_E$ is given by $\overline{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$

Suppose $\mathcal{B}\cup\mathcal{C}_i = \mathcal{F}_i = \left(F_{1i}, F_i, \overline{F}_i\left(\mu_{1F_{1i}}, \mu_{2F_i}\right), L_{F_i}\right)$, where

(7) $F_{1i} = B_1 \cup C_{1i}\, , F_i = B \cap C_i$

(8) $L_{F_i} = L_B \vee L_{C_i}$

(9) $\mu_{1F_{1i}}: F_{1i} \to L_{F_i}$ is given by $\mu_{1F_{1i}}x = \left(\mu_{1B_1} \vee \mu_{1C_{1i}}\right)x$

$\mu_{2F_i}: F_i \to L_{F_i}$ is given by$\mu_{2F_i}x = \left(\mu_{2B} \wedge \mu_{2C_i}\right)x$

$\overline{F}_i: F_i \to L_{F_i}$ is given by$\overline{F}_i x = \mu_{1F_{1i}}x \wedge \left(\mu_{2F_i}x\right)^c$

Let

$\bigcap_{i\in I}(\mathcal{B}\cup\mathcal{C}_i)=\bigcap_{i\in I}\mathcal{F}_i=\mathcal{G}=\left(G_1,G,\overline{G}\left(\mu_{1G_1},\mu_{2G}\right),L_G\right)$, where

(10) $G_1=\bigcap_{i\in I}F_{1i}=\bigcap_{i\in I}(B_1\cup C_{1i})$, $G=\bigcup_{i\in I}(B\cap C_i)$

(11) $L_G=\wedge_{i\in I}L_{F_i}=\wedge_{i\in I}(L_B\vee L_{C_i})$

(12) $\mu_{1G_1}:G_1\to L_G$ is defined by $\mu_{1G_1}x=\left(\wedge_{i\in I}\mu_{1F_{1i}}\right)x=\left[\wedge_{i\in I}\left(\mu_{1B_1}\vee\mu_{1C_{1i}}\right)\right]x$

$\mu_{2G}:G\to L_G$ is defined by $\mu_{2G}x=\left(\vee_{i\in I}\mu_{2F_i}\right)x=\left[\vee_{i\in I}\left(\mu_{2B}\wedge\mu_{2C_i}\right)\right]x$

$\overline{G}:G\to L_G$ is defined by $\overline{G}x=\mu_{1G_1}x\wedge\left(\mu_{2G}x\right)^c$

Need to show that $\mathcal{G}=\bigcap_{i\in I}(\mathcal{B}\cap\mathcal{C}_i)$ exist i.e. Sufficient to show that

(III) $G_1=\bigcap_{i\in I}(B_1\cup C_{1i})\supseteq\bigcup_{i\in I}(B\cap C_i)=G$

(IV) $\mu_{1G_1}x\ge\mu_{2G}x$, for each $x\in G=\bigcup_{i\in I}(B\cap C_i)$

(III) follows from (1) and (4)

Sufficient to show $\mu_{1G_1}x\ge\mu_{2G}x$, for each $x\in G=\bigcup_{i\in I}(B\cap C_i)$

Let $J=\{i\in I|x\in C_i\}\subseteq I$

$$\mu_{2G}x=\bigvee_{j\in J}\mu_{2F_j}x=\bigvee_{j\in J}\left(\mu_{2B}\wedge\mu_{2C_j}\right)x$$
$$=\bigvee_{j\in J}\left(\mu_{2B}x\wedge\mu_{2C_j}x\right)$$
$$=\mu_{2B}x\wedge\left(\bigvee_{j\in J}\mu_{2C_j}x\right)$$

Let $J_1=\{j\in I|x\in C_{1j}\}\subseteq I$, clearly $J\subseteq J_1$

$G_1=\bigcap_{i\in I}F_{1i}=\bigcap_{i\in I}(B_1\cup C_{1i})$, $x\in B\Rightarrow x\in B_1, x\in C_{1j}, j\in J_1$

$$\mu_{1G_1}x=\mu_{1B_1}x\vee\left(\bigwedge_{j\in J}\mu_{1C_{1j}}x\right)$$
$$\mu_{2G}x=\mu_{2B}x\wedge\left(\bigvee_{j\in J}\mu_{2C_j}x\right)\le\mu_{2B}x\le\mu_{1B_1}x$$
$$\le\mu_{1B_1}x\vee\left(\bigwedge_{j\in J}\mu_{1C_{1j}}x\right)=\mu_{1G_1}x$$

Hence $\mathcal{G}=\bigcap_{i\in I}(\mathcal{B}\cap\mathcal{C}_i)$ exist

Need to show $\mathcal{E}=\mathcal{G}$

From (4),(10) and (5),(11) Clearly

(13) $E_1=G_1, E=G$

(14) $L_E=L_G$

Sufficient to show $\mu_{1E_1}x=\mu_{1G_1}x, \mu_{2E}x=\mu_{2G}x$

Case(1) $x\in B_1$ and $x\notin\bigcap_{i\in I}C_{1i}$

$$\mu_{1E_1}x=\mu_{1B_1}x$$
$$\mu_{1G_1}x=\bigwedge_{i\in I}\left(\mu_{1B_1}\vee\mu_{1C_{1i}}\right)x=\bigwedge_{i\in I}\mu_{1B_1}x=\mu_{1B_1}x$$

Hence $\mu_{1E_1}x=\mu_{1G_1}x$

Case(2) $x\notin B_1$ and $x\in\bigcap_{i\in I}C_{1i}$

$$\mu_{1E_1}x=\bigwedge_{i\in I}\mu_{1C_{1i}}x$$
$$\mu_{1G_1}x=\bigwedge_{i\in I}\left(\mu_{1B_1}\vee\mu_{1C_{1i}}\right)x=\bigwedge_{i\in I}\mu_{1C_{1i}}x$$

Hence $\mu_{1E_1}x=\mu_{1G_1}x$

Case(3) $x\in B_1$ and $x\in\bigcap_{i\in I}C_{1i}$

$$\mu_{1E_1}x=\mu_{1B_1}x\vee\left(\bigwedge_{i\in I}\mu_{1C_{1i}}x\right)$$
$$\mu_{1G_1}x=\bigwedge_{i\in I}\left(\mu_{1B_1}x\vee\mu_{1C_{1i}}x\right)=\mu_{1B_1}x\vee\left(\bigwedge_{i\in I}\mu_{1C_{1i}}x\right)$$

Hence $\mu_{1E_1}x=\mu_{1G_1}x$

For $x\in B$ and $x\in C_j$ for all $j\in J$ $x\in C_j,\forall i\in I-J$

$$\mu_{2E}x=\left[\mu_{2B}\wedge\left(\bigvee_{i\in I}\mu_{2C_i}\right)\right]x=\mu_{2B}x\wedge\left(\bigvee_{i\in I}\mu_{2C_i}x\right)$$
$$\mu_{2G}x=\left[\bigvee_{i\in I}\left(\mu_{2B}\wedge\mu_{2C_i}\right)\right]x$$
$$=\left[\bigvee_{j\in J}\left(\mu_{2B}\wedge\mu_{2C_j}\right)x\right]$$
$$\vee\left[\bigvee_{i\in I-J}\left(\mu_{2B}\wedge\mu_{2C_i}\right)x\right]$$
$$=\bigvee_{j\in J}\left(\mu_{2B}x\wedge\mu_{2C_j}x\right)$$
$$=\mu_{2B}x\wedge\left(\bigvee_{i\in I}\mu_{2C_i}x\right)$$

Hence $\mu_{2E}=\mu_{2G}$

Hence $\mathcal{B}\cup(\bigcap_{i\in I}\mathcal{C}_i)=\bigcap_{i\in I}(\mathcal{B}\cup\mathcal{C}_i)$

Proof(B)

Case (I) For $I=\Phi$, $\bigcup_{i\in I}\mathcal{C}_i=\Phi_{\mathcal{A}}$

LHS: $\mathcal{B}\cap(\bigcup_{i\in I}\mathcal{C}_i)=\Phi_{\mathcal{A}}$, RHS: $\bigcup_{i\in I}(\mathcal{B}\cap\mathcal{C}_i)=\Phi_{\mathcal{A}}$

Hence LHS = RHS

Case(II) For each $i\in I$, Let $\mathcal{B}\cap\mathcal{C}_i=\mathcal{F}_i=\left(D_{1i},D_i,\overline{D}_i\left(\mu_{1D_{1i}},\mu_{2D_i}\right),L_{D_i}\right)$, where

(1) $D_{1i}=B_1\cap C_{1i}$, $D_i=B\cup C_i$

(2) $L_{D_i}=L_B\wedge L_{C_i}$

(3) $\mu_{1D_{1i}}:D_{1i}\to L_{D_i}$ is given by $\mu_{1D_{1i}}x=\left(\mu_{1B_1}\wedge\mu_{1C_{1i}}\right)x=\mu_{1B_1}x\wedge\mu_{1C_{1i}}x$

$\mu_{2D_i}:D_i\to L_{D_i}$ is given by $\mu_{2D_i}x=\left(\mu_{2B}\vee\mu_{2C_i}\right)x$

$\overline{D}_i:D_i\to L_{D_i}$ is given by $\overline{D}_ix=\mu_{1D_{1i}}x\wedge\left(\mu_{2D_i}x\right)^c$

Also, we have

(I) $D_{1i}=B_1\cap C_{1i}\supseteq B\cup C_i=D_i$

(II) $\mu_{1D_{1i}}x=\mu_{1B_1}x\wedge\mu_{1C_{1i}}x\ge\left(\mu_{2B}\vee\mu_{2C_i}\right)x=\mu_{2D_i}x$, $\forall x\in D_i$

Let

$\bigcup_{i\in I}(\mathcal{B}\cap\mathcal{C}_i)=\bigcup_{i\in I}\mathcal{D}_i=\mathcal{D}=\left(D_1,D,\overline{D}\left(\mu_{1D_1},\mu_{2D}\right),L_D\right)$, where

(4) $D_1 = \cup_{i\in I} D_{1i} = \cup_{i\in I}(B_1 \cap C_{1i})$, $D = \cap_{i\in I}(B \cup C_i)$
(5) $L_D = \vee_{i\in I} L_{D_i} = \vee_{i\in I}(L_B \wedge L_{C_i})$
(6) $\mu_{1D_1}: D_1 \to L_D$ is defined by $\mu_{1D_1}x = (\vee_{i\in I} \mu_{1D_{1i}})x = [\vee_{i\in I}(\mu_{1B_1} \wedge \mu_{1C_{1i}})]x$

$\mu_{2D} : D \to L_D$ is defined by $\mu_{2D}x = (\wedge_{i\in I} \mu_{2D_i})x = [\wedge_{i\in I}(\mu_{2B} \vee \mu_{2C_i})]x$

$\overline{D}: D \to L_D$ is defined by $\overline{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c$

Let $\cup_{i\in I} \mathcal{C}_i = \mathcal{E} = (E_1, E, \overline{E}(\mu_{1E_1}, \mu_{2E}), L_E)$, where
(7) $E_1 = \cup_{i\in I} C_{1i}$, $E = \cap_{i\in I} C_i$
(8) $L_E = \vee_{i\in I} L_{C_i}$
(9) $\mu_{1E_1}: E_1 \to L_E$ is defined by $\mu_{1E_1}x = (\vee_{i\in I} \mu_{1C_{1i}})x$

$\mu_{2E} : E \to L_E$ is defined by $\mu_{2E}x = (\wedge_{i\in I} \mu_{2C_i})x = \wedge_{i\in I} \mu_{2C_i} x$

$\overline{E}: E \to L_E$ is defined by $\overline{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$

Let $\mathcal{B} \cap (\cup_{i\in I} \mathcal{C}_i) = \mathcal{B} \cap \mathcal{E} = \mathcal{F} = (F_1, F, \overline{F}(\mu_{1F_1}, \mu_{2F}), L_F)$
(10) $F_1 = B_1 \cap E_1 = B_1 \cap (\cup_{i\in I} C_{1i}) = \cup_{i\in I}(B_1 \cup C_{1i})$, $F = B \cup E = B \cup (\cap_{i\in I} C_i) = \cap_{i\in I}(B \cup C_i)$
(11) $L_F = L_B \wedge L_E = L_B \wedge (\vee_{i\in I} L_{C_i}) = \vee_{i\in I}(L_B \wedge L_{C_i})$
(12) $\mu_{1F_1}: F_1 \to L_F$ is given by $\mu_{1F_1}x = (\mu_{1B_1} \wedge \mu_{1E_1})x = [\mu_{1B_1} \wedge (\vee_{i\in I} \mu_{1C_{1i}})]x$

$\mu_{2F} : F \to L_F$ is given by $\mu_{2F}x = (\mu_{2B} \vee \mu_{2E})x = [\mu_{2B} \vee (\wedge_{i\in I} \mu_{2C_i})]x$

$\overline{F}: F \to L_F$ is given by $\overline{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c$

Need to show $\mathcal{B} \cap \mathcal{E} = \mathcal{F} = \mathcal{B} \cap (\cup_{i\in I} \mathcal{C}_i)$ exist i.e.
(III) $F_1 = \cup_{i\in I}(B_1 \cup C_{1i}) \supseteq \cap_{i\in I}(B \cup C_i) = F$
(IV) $\mu_{1F_1}x \geq \mu_{2F}x$, $\forall x \in F = \cap_{i\in I}(B \cup C_i)$
(III) follow from (4)
Let $x \in F = \cap_{i\in I}(B \cup C_i)$

$$J_1 = \{j \in J | x \in C_{1j}\}$$
$$I - J = \{j \in J | x \notin C_{1j}\}$$
$$\mu_{1F_1}x = \mu_{1B_1}x \wedge \left(\bigvee_{j\in J} \mu_{1C_{1j}} x\right) = \bigvee_{j\in J} \left(\mu_{1B_1}x \wedge \mu_{1C_{1j}}x\right) \geq \bigvee_{j\in J} \left(\mu_{2B}x \vee \mu_{2C_j}x\right) \geq \bigwedge_{j\in J} \left(\mu_{2B}x \vee \mu_{2C_j}x\right) \geq \bigwedge_{j\in J} \left(\mu_{2B} \vee \mu_{2C_j}\right)x = \mu_{2F}x$$

Hence $\mu_{1F_1}x \geq \mu_{2F}x$, $\forall x \in F = \cap_{i\in I}(B \cup C_i)$
Need to show $\mathcal{F} = \mathcal{D}$

Clearly
(13) $F_1 = D_1, F = D$ from (4) and (12)
(14) $L_F = L_D$ from (5) and (11)
Sufficient to show
(15) $\mu_{1F_1} = \mu_{1D_1}, \mu_{2F} = \mu_{2D}$

$$\mu_{1F_1}x = \left[\mu_{1B_1} \wedge \left(\bigvee_{i\in I} \mu_{1C_{1i}}\right)\right]x = \mu_{1B_1}x \wedge \left(\bigvee_{i\in I} \mu_{1C_{1i}}\right)$$

$= \mu_{1B_1}x \wedge (\vee_{i\in I_x} \mu_{1C_{1i}})x$, when $I_x = \{i \in I | x \in C_{1i}\}$

$$\mu_{1D_1}x = \left[\bigvee_{i\in I}(\mu_{1B_1} \wedge \mu_{1C_{1i}})\right]x$$

$= \vee_{i\in I'_x}(\mu_{1B_1}x \wedge \mu_{1C_{1i}}x)$ where
$I'_x = \{i \in I | x \in C_{1i}\}$, clearly $I_x = I'_x$

$$= \bigvee_{i\in I_x}(\mu_{1B_1}x \wedge \mu_{1C_{1i}}x) = \mu_{1B_1}x \wedge \left(\bigvee_{i\in I_x} \mu_{1C_{1i}}x\right)$$

Hence $\mu_{1F_1}x = \mu_{1D_1}x$, $\forall x \in F_1$

$$\mu_{2F}x = \left[\mu_{2B} \vee \left(\bigwedge_{i\in I} \mu_{2C_i}\right)\right]x$$

Case(1) $x \in B, x \notin \cap_{i\in I} C_i$

$$\mu_{2F}x = \mu_{2B}x$$
$$\mu_{2D}x = \left[\bigwedge_{i\in I}(\mu_{2B} \vee \mu_{2C_i})\right]x = \bigwedge_{i\in I}(\mu_{2B} \vee \mu_{2C_i})\,x$$

$= \wedge_{i\in I_x}(\mu_{2B} \vee \mu_{2C_i})x \wedge [\wedge_{i\in I-I_x}(\mu_{2B} \vee \mu_{2C_i})x]$, where $I - I_x = \{i \in I | x \notin C_i\}$

$$= \bigwedge_{i\in I_x}(\mu_{2B} \vee \mu_{2C_i})x \wedge (\mu_{2B}x) \quad (\because (\mu_{2B} \vee \mu_{2C_i})x = \mu_{2B}x \text{ where } i \in I - I_x)$$
$$= \left[\mu_{2B}x \vee \left(\bigwedge_{i\in I_x} \mu_{2C_i}x\right)\right] \wedge (\mu_{2B}x)$$

$= [\mu_{2B}x \vee \alpha] \wedge (\mu_{2B}x)$, where $\alpha = \wedge_{i\in I_x} \mu_{2C_i}x$

$$= \mu_{2B}x$$

Clearly $\mu_{2F} = \mu_{2D}$
Case(2): $x \notin B, x \in \cap_{i\in I} C_i$

$$\mu_{2F}x = \left(\bigwedge_{i\in I} \mu_{2C_i}\right)x = \bigwedge_{i\in I} \mu_{2C_i} x$$
$$\mu_{2D}x = \left[\bigwedge_{i\in I}(\mu_{2B} \vee \mu_{2C_i})\right]x = \bigwedge_{i\in I}(\mu_{2B} \vee \mu_{2C_i})\,x = \bigwedge_{i\in I} \mu_{2C_i} x$$

Clearly $\mu_{2F} = \mu_{2D}$
Case(3): $x \in B, x \in \cap_{i\in I} C_i$

$$\mu_{2F}x = \left[\mu_{2B} \vee \left(\bigwedge_{i\in I} \mu_{2C_i}\right)\right]x = \mu_{2B}x \vee \left(\bigwedge_{i\in I} \mu_{2C_i}\right)x = \mu_{2B}x \vee \left(\bigwedge_{i\in I} \mu_{2C_i}x\right)$$

$\mu_{2D}x = [\wedge_{i\in I}(\mu_{2B} \vee \mu_{2C_i})]x = \wedge_{i\in I}(\mu_{2B} \vee \mu_{2C_i})\, x =$
$\wedge_{i\in I}(\mu_{2B}x \vee \mu_{2C_i}x) = \mu_{2B}x \vee (\wedge_{i\in I}\, \mu_{2C_i}x)$

$\mu_{2F} = \mu_{2D}$
Clearly,

Acknowledgements: This paper is dedicated to Dr.M.V.V.S.Murthi, (Ex-MP Govt. of India, MLC of A.P Government) the President , GITAM University,Visakhapatnam-530045,A.P State, India, whose heart throbs for the welfare of the world in terms of high quality academics.

References:

1. Vaddiparthi Yogesara, G.Srinivas and Biswajit Rath ,A Theory of Fs-sets, Fs-Complements and Fs-De Morgan Laws,IJARCS,Vol- 4, No. 10, Sep-Oct 2013
2. *Tazid. Ali, Chandra Kanta Phukan, T*opology in Genetic Code Algebra; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 179-182
3. Vaddiparthi Yogesara, Biswajit Rath and S.V.G.Reddy, A Study of Fs-Functions and Properties of Images of Fs-Subsets Under Various Fs-Functions. MS-IRJ, Vol-3,Issue-1
4. Vaddiparthi Yogesara, Biswajit Rath,A Study of Fs-Functions and Study of Images of Fs-Subsets In The Light Of Refined Definition Of Images Under Various Fs-Functions. IJATCSE, Vol-3, No.3, Pages : 06 - 14 (2014) Special Issue of ICIITEM 2014 - Held during May 12-13, 2014 in PARKROYAL on Kitchener Road, Singapore
5. Tridiv Jyoti Neog and Dusmanta Kumar Sut ,Complement of an Extended Fuzzy Set, IJCA, Vol-29 No.3, September 2011
6. *A. Uma Maheswari, S. Krishnaveni,* Some Properties of Fuzzy Sets on Finite Type of Kac-Moody ... ; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 1 (2013), Pg 39-44
7. Tridiv Jyoti Neog and Dusmanta Kumar Sut, An Extended Approach to Generalized Fuzzy Soft Sets, IJEIC, Vol. 3, Issue 2, May, 2012
8. Hemanta K. Baruah, Towards Forming A Field Of Fuzzy Sets, IJEIC, Vol. 2, Issue 1, February 2011
9. Hemanta K. Baruah, The Theory of Fuzzy Sets: Beliefs and Realities, IJEIC, Vol. 2, Issue 2, May 2011
10. Nistala V.E.S. Murthy, Is the Axiom of Choice True for Fuzzy Sets?, JFM, Vol 5(3),P495-523, 1997, U.S.A
11. Nistala V.E.S Murthy and Vaddiparthi Yogeswara, A Representation Theorem for Fuzzy Subsystems of A Fuzzy Partial Algebra, Fuzzy Sets and System, Vol 104,P359-371,1999,HOLLAND
12. Nistala V.E.S. Murthy, f-Topological Spaces Proceedings of The National Seminar on Topology, Category Theory and their applications to Computer Science, P89-119, March 11-13, 2004, Department of Mathematics, St Joseph's College, Irinjalaguda, Kerala (organized by the Kerala Mathematical Society. Invited Talk).
13. J.A.Goguen ,L-Fuzzy Sets, JMAA,Vol.18, P145-174,1967
14. Manoj Solanki , Kailash Namdeo,Rajesh Shrivastava,Manoj Sharma, Common Fixed Point theorems for S-Weakly; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 87-90
15. Steven Givant • Paul Halmos, Introduction to Boolean algebras, Springer
16. Szasz, G., An Introduction to Lattice Theory, Academic Press, New York.
17. Garret Birkhoff, Lattice Theory, American Mathematical Society Colloquium publications Volume-xxv
18. Thomas Jech , Set Theory, The Third Millennium Edition revised and expanded, Springer
19. George J. Klir and Bo Yuan ,Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A. Zadeh ,Advances in Fuzzy Systems-Applications and Theory Vol-6,World Scientific
20. James Dugundji, Topology, Universal Book Stall, Delhi.
21. L.Zadeh, Fuzzy Sets, Information and Control,Vol.8,P338-353,1965
22. U.Höhle, S.E.Rodabaugh, Mathematics of fuzzy Sets Logic, topology, and measure Theory, Kluwer Academic Publishers, Boston
23. Mamoni Dhar, Fuzzy Sets towards Forming Boolean Algebra, IJEIC, Vol. 2, Issue 4, February 2011
24. Vaddiparthi Yogeswara, Biswajit Rath, Axiom Choice For Fs-sets, communicated

***

Dr. V.Yogeswara/Associate Professor/
Biswajit Rath/Research Scholar/., Ch.Ramasanyasi Rao/Research Scholar/
Dept. Of Mathematics/GITAM University/ Visakhapatnam/ 530045/ A.P State/India.