MEAN SQUARE SUM LABELING OF SOME CYCLE RELATED GRAPHS

C. JAYASEKARAN, S. ROBINSON CHELLATHURAI, M. JASLIN MELBHA

Abstract: A bijection f:$V(G)\rightarrow\{0, 1, ..., p-1\}$ G is said to be a *mean square sum labeling* if the induced function f*:E(G)→N given by f*(uv) = $\left\lfloor\frac{[f(u)]^2+[f(v)]^2}{2}\right\rfloor$or $\left\lceil\frac{[f(u)]^2+[f(v)]^2}{2}\right\rceil$ for every uv $\in E(G)$ is injective. A graph which admits a mean square sum labeling is called a mean square sumgraph. The concept of mean square sum labeling was introduced by C. Jayasekaran, S. Robinson Chellathurai and M. Jaslin Melbha and they investigated the mean square sum labeling of several standard graphs such as Path, Comb, Star graph, Complete graph, Cycle, Bistar, Doublestar, G = K_2+mK_1, Ladder, $P_n \Theta K_2$ and some more graphs are mean square sum graphs. In this paper we prove that Dragon graph, $C_n \Theta K_2$, $C_n \Theta \overline{K}_2$, Helm graph, Wheel graph, Crown graph, Gear graph, nK_3, nC_5 and$D_n \Theta K_1$ are mean square sum graphs.

Keywords: Labeling, mean square sum labeling, mean square sumgraph.

1. Introduction: We begin with simple, finite, connected and undirected graph. For standard terminology and notations we follow Harary [1]. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a *vertex labeling* (an *edge labeling*). Several types of graph labeling and a detailed survey is available in [2]. S. Somasundaram and R. Ponraj [4] have introduced the notion of mean labeling of graphs. A graph G with p vertices and q edges is called *mean graph*if there is an injective function *f* from the vertices of G to {0, 1, ..., q} such that when each edge uv is labeled with $\frac{f(u)+f(v)}{2}$if f(u)+f(v) is even and with $\frac{f(u)+f(v)+1}{2}$ if f(u)+f(v) is odd, then the resulting edge labels are distinct. S. Somasundaram and R. Ponraj [5] have investigated many results on this concept.

V. Ajitha, S. Arumugam and K. A. Germina [6] have introduced the notion of square sum labeling. A (p, q) graph G is said to be square sum, if there exists a bijection f : V(G)→{0, 1, ..., p-1} such that the induced function f*: E(G)→N defined by f*(uv) = $[f(u)]^2 + [f(v)]^2$for every uv $\in E(G)$ is injective.

The concept of mean square sum labeling was introduced by C. Jayasekaran, S. Robinson Chellathurai and M. Jaslin Melbha [3] and they investigated the mean square sum labeling of several standard graphs such as Path, Comb, Star graph, Complete graph, Cycle, Bistar, Doublestar, G = K_2+mK_1, Ladder, $P_n \Theta K_2$. Not every graph is mean square sum. For example, any complete graph K_n, where n ≥ 6 is not mean square sum. We are interested to study different classes of graphs, which are mean square sum. In this paper we prove that Dragon graph, $C_n \Theta K_2$, $C_n \Theta \overline{K}_2$, Helm graph, Wheel graph, Crown graph, Gear graph, nK_3, nC_5 and $D_n \Theta K_1$ are mean square sum graphs.

A brief summary of definitions and other information which are necessary for the present investigation are given below.

Definition 1.1. Let G = $(V(G), E(G))$be a graph. A bijection f :$V(G)\rightarrow\{0, 1, ..., p-1\}$ is said to be a *mean square sum labeling* if the induced function f*: E(G)→N given by f*(uv) = $\left\lfloor\frac{[f(u)]^2+[f(v)]^2}{2}\right\rfloor$or $\left\lceil\frac{[f(u)]^2+[f(v)]^2}{2}\right\rceil$for every uv $\in E(G)$ is injective.

Definition 1.2. A graph which satisfies the mean square sum labeling is called a *mean square sum* graph.

Definition 1.3.A *Dragon* is formed by joining an end point of a path P_m to a point of cycle C_n. It is denoted by $D_n(m)$.

Definition 1.4. A *Helm* H_n, n ≥3 is the graph obtained from a crown by adding a new vertex joined to every vertex of the unique cycle of the crown.

Definition 1.5. The graph $W_n = C_{n-1}+K_1$ is called a *Wheel* with n spokes.A wheel graph W_n is obtained from a cycle C_n by adding a new vertex and joining it to all the vertices of the cycle by an edge, the new edges are called the spokes of the wheel.

Definition 1.6. A *crown* graph is formed by adding to the n points $v_1, v_2, \ldots, v_n$ of a cycle C_n , n more pendent points $u_1, u_2, \ldots, u_n$ and n more lines u_iv_i , i = 1, 2, . . ., n for n ≥3.

Definition 1.7. The *gear* graph is obtained from a wheel by subdividing all the cyclic edges.

Definition 1.8. The corona of two graphs G_1 and G_2 is the graph G = $G_1 \Theta G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where i^{th} vertex of G_1 is adjacent to every vertices in the i^{th} copy of G_2.

Definition 1.9. The prism D_n, n ≥ 3 is a trivalent graph which can be defined as the Cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices.

2. Main Results

Theorem 2.1. The dragon graph $D_{n(m)}$ admits a mean square sum labeling for n ≥3, m ≥1.

Proof. Let $u_1u_2...u_nu_1$ be the cycle C_n and $u_{n+1}, u_{n+2}, ..., u_{n+m}$ be the path P_m. Join the end point u_{n+1} of the path P_m to the cycle C_n. The resultant graph G is a dragon graph with $V(G) = \{u_i /1\leq i\leq n+m\}$ and $E(G) = \{u_iu_{i+1}, u_nu_1/1\leq i\leq n+m\}$. Then G has n+m vertices and n+m edges. Define $f:V(G)\rightarrow\{0, 1, ..., n+m-1\}$ by $f(u_i) = i-1, 1\leq i\leq n+m$. The induced function $f^*:E(G)\rightarrow N$ is defined by $f^*(u_iu_{i+1}) = i^2- i+1, 1\leq i\leq n+m-1$; $f^*(u_nu_1) = \left\lfloor\frac{(n-1)^2}{2}\right\rfloor$ is injective. Hence the dragon graph admits a mean square sum labeling.

Example 2.2. A mean square sum labeling of $D_{4(3)}$ is given in figure 1.

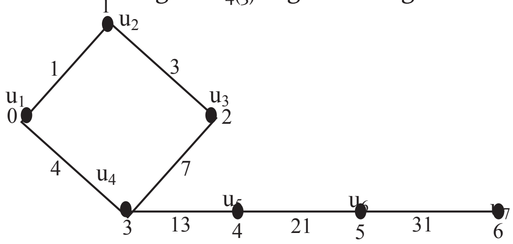

Fig 1. $D_{4(3)}$

Theorem 2.3. $C_n\odot K_2$ is a mean square sum graph.

Proof. Let $u_1, u_1, ..., u_n$ be the cycle C_n and let v_i, w_i be the vertices of i^{th} copy of K_2 which are joined to the vertex u_i of cycle $C_n, 1\leq i\leq n$. The resultant graph is $G = C_n\odot K_2$ with $V(G) = \{u_i, v_i, w_i /1\leq i\leq n\}$ and $E(G) = \{u_iu_{i+1}, u_iv_i, u_iw_i, v_iw_i, u_nu_1, u_nw_n, u_nv_n, v_nw_n/1\leq i\leq n-1\}$.Then G has 3n vertices and 4n edges. Define $f:V(G)\rightarrow\{0, 1, ..., 3n-1\}$ by $f(u_i)=3i-3, 1\leq i\leq n$; $f(v_i)=3i-2, 1\leq i\leq n$; $f(w_i)=3i-1, 1\leq i\leq n$. The induced function $f^*: E(G)\rightarrow N$ is defined by $f^*(u_iw_i) = 9i^2-12i+5, 1\leq i\leq n$; $f^*(v_iw_i) = 9i^2-9i+3, 1\leq i\leq n$; $f^*(u_iv_i) = 9i^2-15i+7, 1\leq i\leq n$; $f^*(u_iu_{i+1}) = 9i^2-9i+5, 1\leq i\leq n-1$; $f^*(u_nu_1) = \left\lceil\frac{9(n-1)^2}{2}\right\rceil$ is injective. Hence $C_n\odot K_2$ is a mean square sum graph.

Example 2.4. A mean square sum labeling of $C_5\odot K_2$ is given in figure 2.

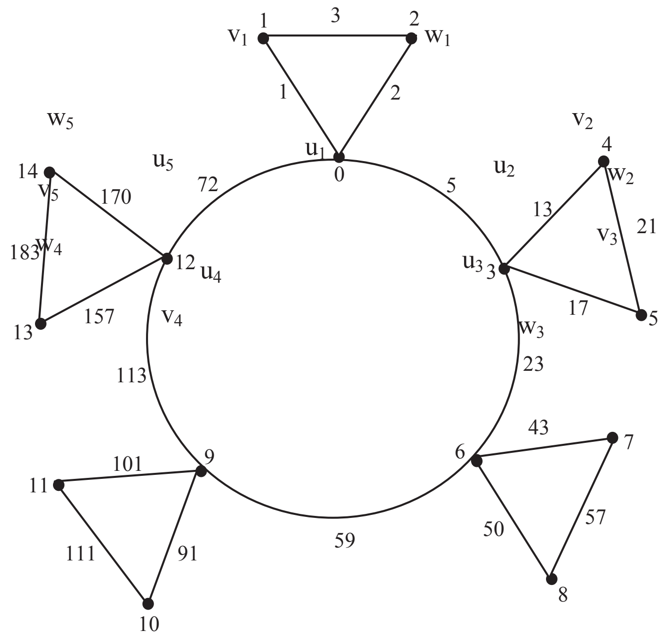

Fig 2.$C_5\odot K_2$

Theorem 2.5. Crown graph $C_n\odot K_1$ is a mean square sum graph for $n\geq 3$.

Proof. Let $u_1u_2...u_nu_1$ be the cycle C_n. For $1\leq i\leq n$, add a new vertex v_i, which is adjacent to u_i. The resultant graph G is the crown graph $C_n\odot K_1$ with $V(G) = \{u_i, v_i /1\leq i\leq n\}$ and $E(G) = \{u_iu_{i+1}, u_nu_1, u_iv_i, v_nu_n /1\leq i\leq n-1\}$. Then G has 2n vertices and 2n edges. Define $f:V(G)\rightarrow\{0, 1, ..., 2n-1\}$ by $f(u_i) = i-1, 1\leq i\leq n$ and $f(v_i) = n+i-1, 1\leq i\leq n$. The induced function $f^*: E(G)\rightarrow N$ is defined by $f^*(u_iu_{i+1}) = i^2-i+1, 1\leq i\leq n-1$; $f^*(u_nu_1) = \left\lfloor\frac{(n-1)^2}{2}\right\rfloor$ and $f^*(u_iv_i) = \left\lceil\frac{n^2}{2} + n(i-1) + (i-1)^2\right\rceil$, $1\leq i\leq n$ is injective. Hence the crown graph is a mean square sum graph.

Example 2.6. A mean square sum labeling of $C_7\odot K_1$ is given in figure 3.

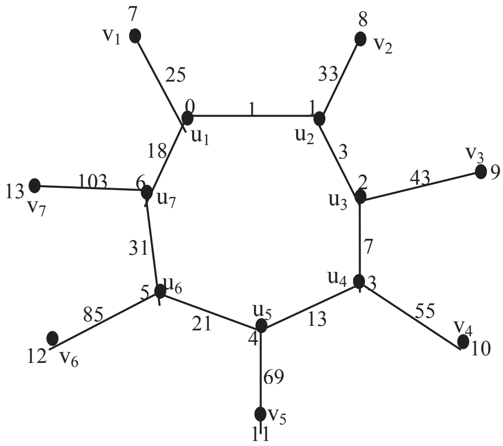

Fig 3. $C_7 \odot K_1$

Theorem 2.7. The Helm H_n is a mean square sum graph for $n \geq 3$.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of C_n. For $1 \leq i \leq n$, add a new vertex v_i, which is adjacent to u_i. The resultant graph is crown graph. By adding a new vertex u joined to every vertex of the unique cycle of the crown. The resultant graph is the Helm graph H_n with $V(H_n) = \{u, u_1, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and $E(H_n) = \{uu_i, uu_n, u_iu_{i+1}, u_nu_1, u_iv_i, v_nu_n / 1 \leq i \leq n-1\}$. Then H_n has 2n+1 vertices and 3n edges. Define $f : V(H_n) \rightarrow \{0, 1, \ldots, 2n\}$ by $f(u) = 0$; $f(u_i) = i$, $1 \leq i \leq n$ and $f(v_i) = n+i$, $1 \leq i \leq n$. The induced function $f^*: E(H_n) \rightarrow N$ is defined by $f^*(u_iu_{i+1}) = i^2+i+1$, $1 \leq i \leq n-1$; $f^*(uu_1) = 1$; $f^*(uu_i) = \left\lfloor \frac{i^2}{2} \right\rfloor$, $2 \leq i \leq n$; $f^*(u_nu_1) = \left\lceil \frac{n^2+1}{2} \right\rceil$ and $f^*(u_iv_i) = \frac{n^2}{2} + i + i^2$, $1 \leq i \leq n$ is injective for $n \neq 5$. For $n = 5$, $\frac{(0^2+5^2)}{2} = \frac{(3^2+4^2)}{2} = 12.5$ and $\frac{(5^2+1^2)}{2} = 13$. This implies that at least two edges get the same label. H_5 is a mean square sum graph and a labeling is given in the figure 4. Hence Helm graph H_n is a mean square sum graph.

Example 2.8. A mean square sum labeling of H_6 is given in figure 4.

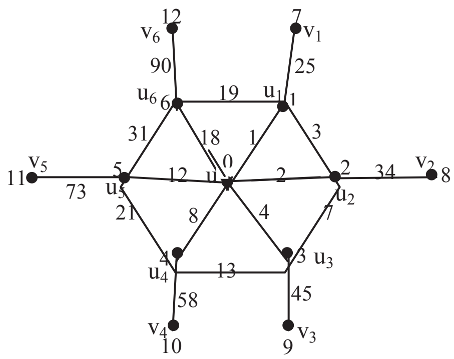

Fig 4. H_6

Example 2.9. A mean square sum labeling of H_5 is given in figure 5.

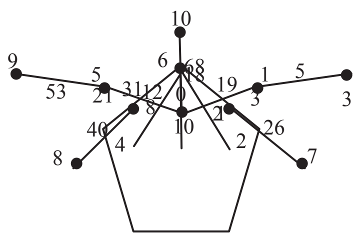

Fig 5. H_5

Theorem 2.10. The Wheel graph W_n admits a mean square sum graphs for $n \geq 3$.

Proof . Let $u_1, u_2, \ldots, u_n$ be the vertices of C_n. By adding a new vertex u joined to every vertex of the unique cycle by an edge, the resultant graph is the wheel W_n with $V(W_n) = \{u, u_1, \ldots, u_n\}$ and $E(W_n) = \{uu_i, uu_n, u_iu_{i+1}, u_nu_1 / 1 \leq i \leq n-1\}$. Then W_n has n+1 vertices and 2n edges. Define $f: V(W_n) \rightarrow \{0, 1, \ldots, n\}$ by $f(u) = 0$;

$f(u_i) = i$ for $1 \le i \le n$. The induced function $f^*: E(W_n) \to N$ is defined by $f^*(u_iu_{i+1}) = i^2+i+1$, $1 \le i \le n-1$; $f^*(uu_1) = 1$; $f^*(uu_i) = \left\lfloor \frac{i^2}{2} \right\rfloor$, $2 \le i \le n$; $f^*(u_1u_n) = \left\lceil \frac{n^2+1}{2} \right\rceil$ is injective for $n \neq 5$. For $n = 5$, $\frac{(0^2+5^2)}{2} = \frac{(3^2+4^2)}{2} = 12.5$ and $\frac{(5^2+1^2)}{2} = 13$. This implies that atleast two edges get the same label. W_5 is a mean square sum graph and a labeling is given in the figure 6. Hence wheel graph is a mean square sum graph.

Example 2.11. A mean square sum labeling of W_6 is given in figure 6.

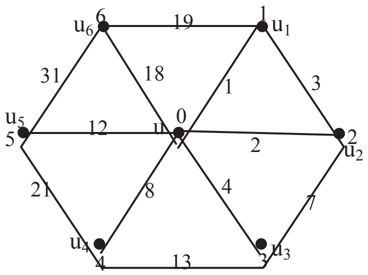

Fig 6. W_6

Example 2.12. A mean square sum labeling of W_5 is given in figure 7.

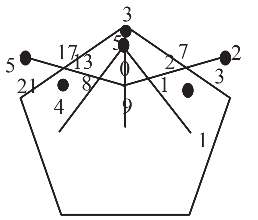

Fig 7. W_5

Theorem 2.13. nK_3 is a mean square sum graph.

Proof. Let v_{i1}, v_{i2}, v_{i3} be the vertices of i^{th} copy of K_3. Let $G = nK_3$. Then $V(G) = \{v_{ij} / 1 \le i \le n; 1 \le j \le 3\}$ and $E(G) = \{v_{ij}v_{i(j+1)}, v_{i3}v_{i1} / 1 \le i \le n, 1 \le j \le 2\}$. Then G has 3n vertices and 3n edges. Define $f: V(G) \to \{0, 1, \ldots, 3n-1\}$ by $f(v_{ij}) = 3i+j-4$, $1 \le i \le n$, $1 \le j \le 3$. The induced function $f^*: E(G) \to N$ is defined by $f^*(v_{ij}v_{ij+1}) = 9i^2+j^2-21i-7j+6ij+13$, $1 \le i \le n$, $1 \le j \le 2$ and $f^*(v_{i3}v_{i1}) = 9i^2-12i+5$, $1 \le i \le n$ is injective. Hence nK_3 is a mean square sum graph.

Example 2.14. A mean square sum labeling of $4K_3$ is given in figure 8.

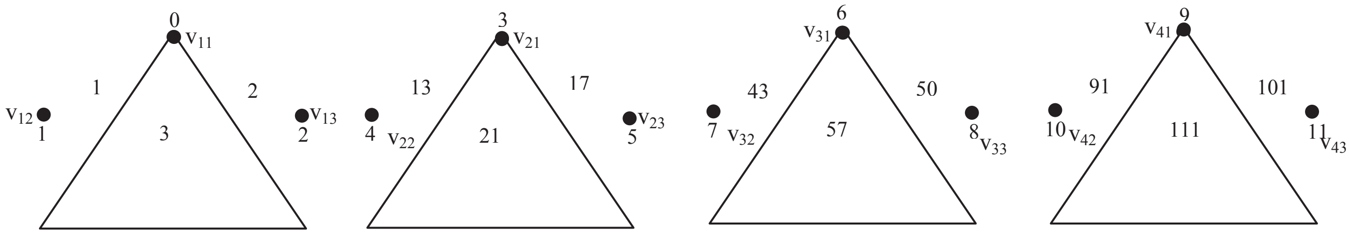

Fig 8. $4K_3$

Theorem 2.15. nC_5 is a mean square sum graph.

Proof. Let $v_{5i+1}, v_{5i+2}, v_{5i+3}, v_{5i+4}, v_{5i+5}$ be the vertices of $(i+1)^{th}$ copy of C_5 where $i = 0, 1, \ldots, n-1$. Let $G = nC_5$. Then $V(G) = \{v_{5i+1}, v_{5i+2}, v_{5i+3}, v_{5i+4}, v_{5i+5} / i = 0, 1, \ldots, n-1\}$ and $E(G) = \{v_{5i+j}v_{5i+j+1}, v_{5i+5}v_{5i+1} / 0 \le i \le n-1, 1 \le j \le 4\}$. Then G has 5n vertices and 5n edges. Define $f: V(G) \to \{0, 1, \ldots, 5n-1\}$ by $f(v_{5i+j}) = 5i+j-1$, $0 \le i \le n-1$, $1 \le j \le 5$. The induced function $f^*: E(G) \to N$ is defined by $f^*(v_{5i+j}v_{5i+j+1}) = 25i^2+j^2+10ij-5i-j+1$, $0 \le i \le n-1$, $1 \le j \le 5$ and $f^*(v_{5i+5}v_{5i+1}) = 25i^2+20i+8$, $0 \le i \le n-1$ is injective. Hence nC_5 is a mean square sum graph.

Example 2.16. A mean square sum labeling of $4C_5$ is given in figure 9.

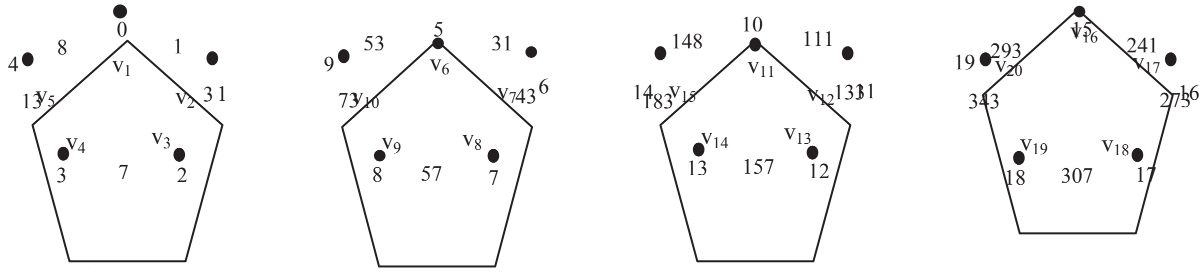

Fig 9. $4C_5$

Theorem 2.17. The gear graph G_n is a mean square sum graph.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of C_n. By adding a new vertex u joined to every vertex of the unique cycle by an edge, we get the wheel W_n. Subdivide the edge u_iu_{i+1} in to two edges u_iv_i and v_iu_{i+1}, $1\le i\le n-1$ and the edge u_nu_1 into the edges u_nv_n and v_nu_1. The resultant graph is the gear graph G_n with $V(G_n) = \{u, u_1, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and $E(G_n) = \{uu_i, u_iv_i, v_iu_{i+1}, uu_n, u_nv_n, v_nu_1 / 1\le i\le n-1\}$. Then G_n has 2n+1 vertices and 3n edges. Define f: $V(G_n) \rightarrow \{0, 1, \ldots, 2n\}$ by f(u) = 0; $f(u_i) = 2i-1$, $1\le i\le n$ and $f(v_i) = 2i$, $1\le i\le n$. The induced function f*: $E(G_n)\rightarrow N$ is defined by $f^*(v_iu_{i+1}) = 4i^2+2i+1$, $1\le i\le n-1$; $f^*(uu_i) = \left\lceil\frac{i^2}{2}\right\rceil$, $1\le i\le n$; $f^*(u_iv_i) = 4i^2-2i$, $1\le i\le n$; $f^*(u_1v_n) = \left\lceil\frac{(2n)^2+1}{2}\right\rceil$ is injective. Hence the gear graph G_n is a mean square sum graph.

Example 2.18. A mean square sum labeling of G_5 is given in figure 10.

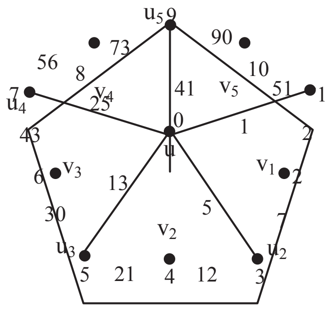

Fig 10. G_5

Theorem 2.19. $C_n \Theta \overline{K}_2$ is a mean square sum graph for all $n \ge 3$.

Proof. Let $u_1u_2\ldots u_nu_1$ be the cycle C_n and v_i, w_i be the pendant vertices adjacent to u_i, $1\le i\le n$. The resultant graph $G = C_n\Theta\overline{K}_2$ with $V(G) = \{u_i, w_i, v_i /1\le i\le n\}$ and $E(G) = \{u_nu_1, u_iv_i, u_iw_i, u_ju_{j+1}/ 1\le j\le n-1, 1\le i\le n\}$. Then G has 3n vertices and 3n edges. Define f:$V(G)\rightarrow\{0, 1, \ldots, p-1\}$ by $f(u_i) = 3i-3$, $f(w_i) = 3i-2$, and $f(v_i) = 3i-1$, for $1\le i\le n$. The induced function f*: $E(G)\rightarrow N$ is defined by $f^*(u_iw_i) = 9i^2-15i+7$, $1\le i\le n$; $f^*(u_nu_1) = \left\lfloor\frac{(3n-3)^2}{2}\right\rfloor$; $f^*(u_iv_i) = 9i^2-12i+5$, $1\le i\le n$; $f^*(u_iu_{i+1}) = 9i^2-9i+5$, $1\le i\le n-1$ is injective. Hence $C_n\Theta\overline{K}_2$ is a mean square sum graph.

Example 2.20. A mean square sum labeling of $C_5\Theta\overline{K}_2$ is given in figure 11.

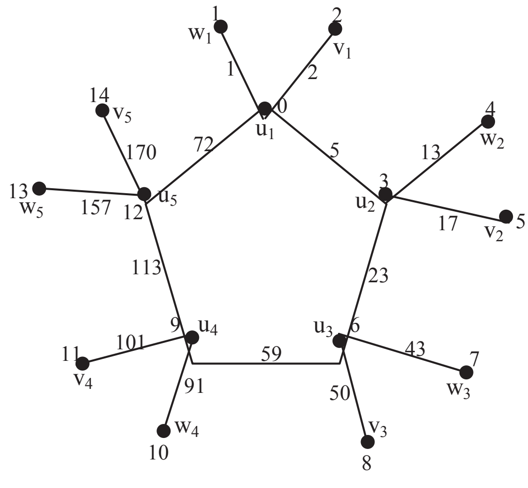

Fig.11. $C_5\Theta\overline{K}_2$

Theorem 2.21. $D_n\Theta K_1$ is a mean square sum graph $n \ge 3$.

Proof. Let $u_1u_2\ldots u_nu_1$ and $v_1v_2\ldots v_nv_1$ be two cycles of length n. Join u_i and v_i, $1\le i\le n$. The resultant graph is D_n (i.e) $P_2\times C_n$. For $1\le i\le n$, let s_i and t_i be the vertices which are joined with u_i and v_i respectively. The resultant graph is $G = D_n\Theta K_1$ with $V(G) = \{u_i, t_i, v_i, s_i /1\le i\le n\}$ and $E(G) = \{u_ju_{j+1}, u_nu_1, v_jv_{j+1}, v_nv_1, u_iv_i, u_is_i, v_it_i / 1\le j\le n-1, 1\le i\le n\}$. Then G has 4n vertices and 5n edges. Define f :$V(G)\rightarrow\{0, 1, \ldots, p-1\}$ by $f(u_i) = 4i-3$, $f(v_i) = 4i-2$, $f(t_i) = 4i-1$, $f(s_i) = 4i-4$ for $1\le i\le n$. The induced function f*: $E(G)\rightarrow N$ is defined by $f^*(u_is_i) = 16i^2-28i+13$, $1\le i\le n$; $f^*(v_iu_i) = 16i^2-20i+7$, $1\le i\le n$; $f^*(v_it_i) = 16i^2-12i+3$, $1\le i\le n$; $f^*(u_iu_{i+1}) = 16i^2-8i+5$, $1\le i\le n-1$; $f^*(v_iv_{i+1}) = 16i^2+4$, $1\le i\le n-1$; $f^*(u_nu_1) = 8n^2-12n+5$; $f^*(v_nv_1) = 8n^2-8n+4$ is injective. Hence $D_n\Theta K_1$ is a mean square sum graph.

Example 2.22. A mean square sum labeling of $D_5\Theta K_1$ is given in figure 12.

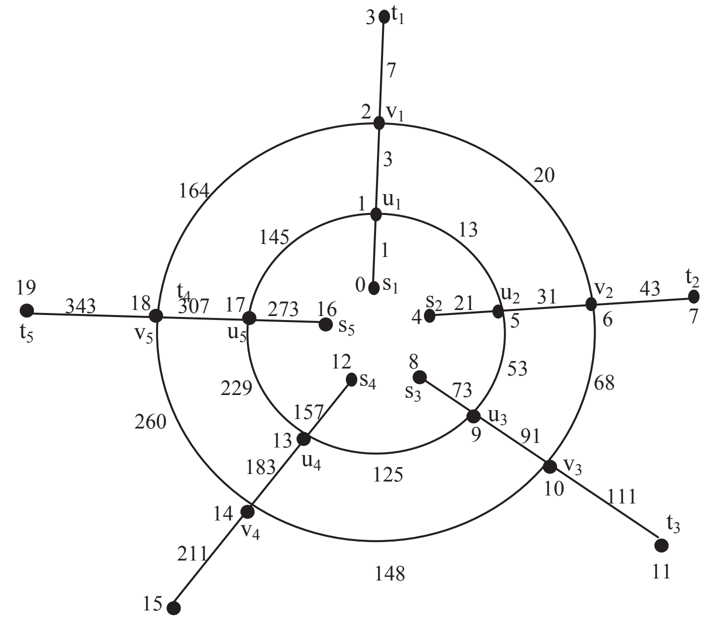

Fig 12. $D_5 \odot K_1$

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