WEIGHTED SHARING OF DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF ENTIRE FUNCTIONS

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Abstract : In this paper, we study the uniqueness of difference-differential polynomials of entire functions f and g sharing one value with weight l. In this paper we extend and generalize the results of L.Kai, L.Xin-ling, C.Ting-bin[6].

Keywords: Nevanlinna theory, Entire functions, Difference-differential polynomials, Weighted Sharing, Uniqueness, etc.

Introduction and Main Results: In this paper, the term meromorphic will always mean meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [5],[10] and [12]. We denote by $S(r,f)$ any function satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to+\infty$, possibly outside of a set with finite linear measure.

Definition 1.1: $\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\dfrac{\overline{N}(r,a,f)}{T(r,f)}$

$$\delta_k(a,f)=1-\overline{\lim_{r\to\infty}}\frac{N_k(r,a,f)}{T(r,f)}$$

where $'a'$ be any complex number and k be a positive integer. We denote $N_k(r,a,f)=\overline{N}_{(1}(r,a,f)+\overline{N}_{(2}(r,a,f)+\ldots+\overline{N}_{(k}(r,a,f)$, where $N_{k)}(r,a,f)$ the counting function for the zeros of $f(z)-a$ with multiplicity $\le k$, and $\overline{N}_{k)}(r,a,f)$ the corresponding one for which the multiplicity is not counted; $N_{(k}(r,a,f)$ be the counting function for the zeros of $f(z)-a$ with multiplicity $\ge k$, and $\overline{N}_{(k}(r,a,f)$ be the corresponding one for which the multiplicity is not counted.

In 2001, Lahiri first introduced the idea of weighted sharing of values which measures how close a shared value is to being shared IM or to being shared CM. Recently, many mathematicians such as H.X.Yi, I.Lahiri, M.L.Fang, A.Banerjee, W.C.Lin, X.Yan have been interested in investigating meromorphi -c functions sharing values with some weight in the field of Nevanlinna theory.

Definition 1.2: Let k be a non-negative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m\le k$ and $k+1$ times if $m>k$. If $E_k(a,f)=E_k(a,g)$, we say that f, g share the value a with weight k.

In 2008, Zhang and Lu[14] proved the following theorem using the concept of weighted sharing.

Theorem A : Let $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions, and let $n(\ge 1),k(\ge 1),l(\ge 0)$ be three integers. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share $(1,l)$, if $l\ge 2$ and $n>3k+8$ or if $l=1$ and $n>5k+11$ or $l=0$ and $9k+14$, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$ or $f(z)\equiv tg(z)$ for a constant t such that $t^n=1$.

In 2012, L.Kai, L.Xin-ling, C.Ting-bin [6] proved the unicity of difference polynomials given below.

Theorem B : Let f and g be transcendental entire functions of finite order, $n\ge 2k+6$ and c is a non-zero complex constant. If $[f^n(z)f(z+c)]^{(k)}$ and $[g^n(z)g(z+c)]^{(k)}$ share the value 1 CM, then either $f(z)=c_1e^{Cz}$, $g(z)=c_2e^{-Cz}$, where c_1,c_2 and C are constants satisfying $(-1)^k(c_1c_2)^{n+1}((n+1)C)^{2k}=1$, or $f\equiv tg$ for a constant t such that $t^{n+1}=1$.

Recently, R.S.Dyavanal and A.M.Hattikal[2] proved the following result concerning to difference-differential polynomials.

Theorem C: Let f and g be two non-constant entire functions of finite order. Let n,k and m be three positive integers with $n\ge m+2k+6$, $'c'$ is a non-zero complex constant and $P(z)=a_mz^m+a_{m-1}z^{m-1}+\ldots+a_1z+a_0$ or $P(z)\equiv c_0$ where $a_0(\ne 0),a_1,a_2,a_3,\ldots,a_{m-1},a_m(\ne 0)$,

$c_0(\neq 0)$ are complex constants. If $[f^n(z)P(f)f(z+c)]^{(k)}$ and $[g^n(z)P(g)g(z+c)]^{(k)}$ share 1 CM,

1. when $P(z)=a_m z^m + a_{m-1}z^{m-1}+\ldots+a_1 z+a_0$ we get $f(z)\equiv tg(z)$ for a constant t such that $t^d=1$, where, $d=GCD\{n+m+1, n+m ,\ldots, n+m+1-i,\ldots, n+1\}$ and $i=0,1,2,\ldots,m$.
2. when $P(z)\equiv c_0$ either $f(z)=\frac{c_1e^{Cz}}{\sqrt[n]{c_0}}$, $g(z)=\frac{c_2e^{-Cz}}{\sqrt[n]{c_0}}$, where c_1,c_2,c_0 and C are constants satisfying $(-1)^k(c_1c_2)^{n+1}((n+1)C)^{2k}=(\sqrt[n]{c_0})^2$, or $f\equiv tg$ for a constant t such that $t^{n+1}=1$.

Question 1.1: What happens if sharing value 1 with counting multiplicity is replac
-ed by weighted sharing in theorem C?

In the present paper we tried to answer the above question by proving the following theorem.

Theorem 1.1: Let f and g be two non-constant entire functions, and let n. m and k be three positive integers. $P(z)$ is as defined in the theorem C. If $[f^nP(f)f(z+c)]^{(k)}$ and $[g^nP(g)g(z+c)]^{(k)}$ share $(1,l)$, with the conditions of n as below.

1. $n\geq 3m+2k+7$, when $l\geq 2$
2. $n\geq 4m+3k+9$, when $l=1$
3. $n\geq 6m+5k+13$, when $l=0$, then

I. when $P(z)=a_m z^m + a_{m-1}z^{m-1}+\ldots+a_1 z+a_0$, we get $f(z)\equiv tg(z)$ for a constant t such that $t^d=1$, where $d=GCD\{n+m+1, n+m,\ldots, n+m+1-i, \ldots, n+1\}$ and $i=0,1,2,\ldots,m$.

II. when $P(z)\equiv c_0$ then either $f(z)=\frac{c_1e^{Cz}}{\sqrt[n]{c_0}}$, $g(z)=\frac{c_2e^{-Cz}}{\sqrt[n]{c_0}}$, where c_1,c_2,c_0 and C are constants satisfying $(-1)^k(c_1c_2)^{n+1}((n+1)C)^{2k}=(\sqrt[n]{c_0})^2$, or $f\equiv tg$ for a constant t such that $t^{n+1}=1$.

2. Some Lemmas

We need following Lemmas to prove our results.

Lemma 2.1 ([1]): Let $f(z)$ be a trancendental meromorphic function of finite order, then

$$T(r,f(z+c))=T(r,f)+S(r,f)$$

Lemma 2.2 ([12]) : Let $f(z)$ be a nonconstant meromorphic function, and $a_n(\neq 0), a_{n-1},\ldots,a_0$ be small functions with respect to f. Then

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\ldots+a_0)=nT(r,f)+S(r,f)$$

Lemma 2.3 ([4]) : Let f be a transcendental meromorphic function of finite order. Then

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=S(r,f)$$

Lemma 2.4 ([1],[3]) : Let $f(z)$ be a meromorphic function of finite order and c is a non-zero complex constant. Then

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=S(r,f)$$

Lemma 2.5 ([8]): Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $k(\geq 1)$, $l(\geq 0)$ be integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1,l)$. If one of the following conditions holds, then either $f^{(k)}g^{(k)}\equiv 1$ or $f(z)\equiv g(z)$.

1. $l\geq 2$ and $\Delta_1=2\Theta(\infty,f)+(k+2)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>k+7$
2. $l=1$ and $\Delta_2=(k+3)\Theta(\infty,f)+(k+2)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>2k+9$
3. $l=0$ and $\Delta_3=(2k+4)\Theta(\infty,f)+(2k+3)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+3\delta_{k+1}(0,f)+2\delta_{k+1}(0,g)>4k+13$

Lemma 2.6 ([2]): Let $f(z)$ be a transcendental entire function of finite order and let $F^*=f(z)^nP(f)f(z+c)$. Then

$$T(r,F^*)=(n+m+1)T(r,f)+S(r,f)$$

Lemma 2.7 ([2]): Let $f(z)$ and $g(z)$ be two nonconstant entire functions of finite order, let n,k be two positive integers with $n>k$, $'c'$ is a non-zero complex constant and let $P(z)=a_mz^m+a_{m-1}z^{m-1}+\ldots+a_1z+a_0$ be a non-zero polynomial, where $a_0,a_1,a_2,\ldots a_{m-1},a_m$ are complex constants. If

$[f^nP(f)f(z+c)]^{(k)}[g^nP(g)g(z+c)]^{(k)} \equiv 1$, then $P(z)$ is reduced to a non-zero monomial, that is $P(z)=a_iz^i \neq 0$ for some $i=0,1,2\dots m$.

Lemma 2.8([2]) : Let $f(z)$ and $g(z)$ be two trancendental entire functions of finite order. Let n,k be two positive integers with $n>m+2k+6$, $'c'$ is a non-zero complex constant and $P(z)=a_mz^m+a_{m-1}z^{m-1}+\dots+a_1z+a_0$ be a non-zero polynomial, where $a_0,a_1,a_2,\dots a_{m-1},a_m$ are complex constants. If $[f^nP(f)f(z+c)]^{(k)} \equiv [g^nP(g)g(z+c)]^{(k)}$, then $f(z)\equiv tg(z)$, such that $t^d=1$, where $d=GCD\{n+m+1,\dots,n+m+1-i,\dots,n+1\}$ and $i=0,1,2,\dots,m$.

Proof of theorem 1.1

I. If $P(z)=a_mz^m+a_{m-1}z^{m-1}+\dots+a_1z+a_0$.

Then by assumption and Theorem 1.1 we know that either both f and g are transcendental entire functions or both f and g are polynomials.

Considering $F=[f^n(z)P(f)f(z+c)]^{(k)}$, $G=[g^n(z)P(g)g(z+c)]^{(k)}$.

Using Lemma 2.6 we have

$\Theta(\infty,F)=1, \Theta(\infty,G)=1,$

$$\Theta(0,F)\geq \frac{n-1}{n+m+1}, \quad \Theta(0,G)\geq \frac{n-1}{n+m+1}$$

$$\delta_{k+1}(0,F)\geq \frac{n-k-1}{n+m+1}, \delta_{k+1}(0,G)\geq \frac{n-k-1}{n+m+1} \quad (1)$$

Now we consider three cases for l as below.

Using (1), we have

Case 1: When $l\geq 2$,

$$\Delta_1 = 2\Theta(\infty,F)+(k+2)\Theta(\infty,G)+\Theta(0,F)+\Theta(0,G)+\delta_{k+1}(0,F)+\delta_{k+1}(0,G)\geq k+7;$$

Case 2: When $l=1$,

$$\Delta_2=(k+3)\Theta(\infty,F)+(k+2)\Theta(\infty,G)+\Theta(0,F)+\Theta(0,G)+2\delta_{k+1}(0,F)+\delta_{k+1}(0,G)\geq 2k+9;$$

Case 3: When $l=0$,

$$\Delta_3=(2k+4)\Theta(\infty,F)+(2k+3)\Theta(\infty,G)+\Theta(0,F)+\Theta(0,G)+3\delta_{k+1}(0,F)+2\delta_{k+1}(0,G)\geq 4k+13.$$

Since F and G share $(1,l)$ and using Lemma 2.5 we get either $FG\equiv 1$ or $F\equiv G$.

Suppose $FG\equiv 1$ holds,

i.e
$$\left[f^n(z)(a_mf^m+\dots+a_0)f(z+c)\right]^{(k)}\left[g^n(z)(a_mg^m+\dots+a_0)g(z+c)\right]^{(k)}\equiv 1 \quad (2)$$

By assumption that $a_m\neq 0, a_0\neq 0,$ we can arrive at a contradiction by Lemma 2.7.

Hence, we have $F(z)\equiv G(z)$, if we consider the case when f and g are transcendental entire functions, by Lemma 2.8 we get conclusion of the theorem 1.1.

Now we consider the case when f and g are two polynomials.

Since $[f^n(z)P(f)f(z+c)]^{(k)}$ and $[g^n(z)P(g)g(z+c)]^{(k)}$ share 1 CM, we have

$$\left[f^n(z)(a_mf^m+\dots+a_0)f(z+c)\right]^{(k)}-1=\beta\left[\left[g^n(z)(a_mg^m+\dots+a_0)g(z+c)\right]^{(k)}-1\right] \quad (3)$$

where β is a non-zero constant. Let $deg\ f=l_0$, then by (3) we know that $deg\ g=l_0$.

Differentiating the two sides of (3), we get

$$f^{n-k-1}(z)q_1(z)=g^{n-k-1}(z)q_2(z), \quad (4)$$

where $q_1(z)$, $q_2(z)$ are two polynomials with $deg\ q_1(z)=deg\ q_2(z)=(m+k+2)l_0-(k+1).$

By the conditions of n, we get $deg\ f^{n-k-1}(z)=(n-k-1)l_0>deg\ q_2(z)$.

Thus, by (4) we know that there exists z_0 such that $f(z_0)=g(z_0)=0$.

Hence, by (3) and $f(z_0)=g(z_0)=0$, we deduce that $\beta=1$, that is,

$$\left[f^n(z)(a_mf^m+\dots+a_0)f(z+c)\right]^{(k)}=\left[g^n(z)(a_mg^m+\dots+a_0)g(z+c)\right]^{(k)} \quad (5)$$

Thus, we have

$$f^n(z)(a_mf^m+\dots+a_0)f(z+c)-g^n(z)(a_mg^m+\dots+a_0)g(z+c)=Q(z) \quad (6)$$

where $Q(z)$ is a polynomial of degree atmost $k-1$.

Next we prove $Q(z)\equiv 0$. By rewriting (5) as

$$f^{n-k}(z)p_1(z)=g^{n-k}(z)p_2(z) \quad (7)$$

where $p_1(z)$, $p_2(z)$ are two polynomials with $deg\ p_1(z)=deg\ p_2(z)=(m+k+1)l_0-k$ and $deg\ f(z)=l_0$.

Hence total number of common zeros of $f^{n-k}(z)$ and $g^{n-k}(z)$ is atleast k. Thus, by (6) we deduce that $Q(z)\equiv 0$, that is

$$f^n(z)(a_m f^m + a_{m-1}f^{m-1} + \ldots + a_0)f(z+c) = g^n(z)(a_m g^m + a_{m-1}g^{m-1} + \ldots + a_0)g(z+c) \quad (8)$$

Next, similar to the argument as in Lemma 2.8, we get $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where

$d = GCD\{n+m+1, \ldots, n+m+1-i, \ldots, n+1\}$

and $i = 0,1,2,\ldots,m.$

Hence proved the I of Theorem 1.1.

II. If $P(z) \equiv c_0$

By the assumption and Theorem 1.1, we know that either both f and g are transcendental entire functions or both f and g are polynomials.

First, we consider the case when both f and g are transcendental entire functions.

$Let \quad F = f^n(z)c_0 f(z+c),$

$G = g^n(z)c_0 g(z+c)$

By the Theorem B and $n \geq 2k+6$, we obtain either $f(z) = \frac{c_1 e^{Cz}}{\sqrt[n]{c_0}}$, $g(z) = \frac{c_2 e^{-Cz}}{\sqrt[n]{c_0}}$, where c_1, c_2, c_0 and C are constants satisfying $(-1)^k (c_1 c_2)^{n+1}((n+1)C)^{2k} = (\sqrt[n]{c_0})^2$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

Now we consider the case when both f and g are two polynomials.

Since $[f^n(z)c_0 f(z+c)]^{(k)}$ and $[g^n(z)c_0 g(z+c)]^{(k)}$ share 1 CM, we have

$$\left[f^n(z)c_0 f(z+c)\right]^{(k)} - 1 = \gamma\left[\left[g^n(z)c_0 g(z+c)\right]^{(k)} - 1\right] \quad (9)$$

where γ is a non-zero constant. Let $deg\, f(z) = l_1$, then by (9)we know that $deg\, g(z) = l_1$. Differentiating the two sides of (9), we get

$$f^{n-k-1}(z)q_3(z) = g^{n-k-1}(z)q_4(z) \quad (10)$$

where $q_3(z)$, $q_4(z)$ are two polynomials with $deg\, q_3(z) = deg\, q_4(z) = (\mathrm{k}+2)\mathrm{l}_1 - (k+1)$ By the conditions of n, we get $deg\, f^{n-k-1}(z) = (n-k-1)l_1 > deg\, q_4(z)$.

Thus, by (10) we know that there exists z_0 such that $f(z_0) = g(z_0) = 0$.

Hence, by (9) and $f(z_0) = g(z_0) = 0$, we deduce that $\gamma = 1$, that is,

$$\left[f^n(z)c_0 f(z+c)\right]^{(k)} = \left[g^n(z)c_0 g(z+c)\right]^{(k)} \quad (11)$$

Thus, we have

$$f^n(z)f(z+c) - g^n(z)g(z+c) = Q_1(z) \quad (12)$$

where $Q_1(z)$ is a polynomial of degree atmost $k-1$.

Next we prove $Q_1(z) \equiv 0$.

By rewriting (11) as

$$f^{n-k}(z)p_3(z) = g^{n-k}(z)p_4(z) \quad (13)$$

where $p_3(z)$, $p_4(z)$ are two polynomials with $deg\, p_3(z) = deg\, p_4(z) = (k+1)l_1 - k$ and $deg\, f(z) = l_1$.

Hence total number of common zeros of $f^{n-k}(z)$ and $g^{n-k}(z)$ is atleast k.

Thus, by (12) we deduce that $Q_1(z) \equiv 0$, that is,

$$f^n(z)f(z+c) = g^n(z)g(z+c) \quad (14)$$

Let $h(z) = \frac{f(z)}{g(z)}$ and $h(z+c) = \frac{f(z+c)}{g(z+c)}$ then

$(gh)^n g(z+c)h(z+c) = g^n g(z+c)$

Hence $f = tg$ where $(tg)^n tg(z+c) = g^n g(z+c)$

$\Rightarrow t^{n+1} = 1$

Hence proved the II of Theorem 1.1.

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References:

1. Y.M Chiang, S.J. Feng, "On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane", Ramanujan J.,16, (2008):105-129.
2. R.S.Dyavanal, A.M.Hattikal, "Uniqueness of difference-differential polynomials of entire functions sharing one value", Tamakang J. Math.(Accepted).
3. R.G.Halburd, R.J.Korhonen, "Nevanlinna theory for the difference operator", Ann. Acad. Sci. Fenn., 31,(2006): 463-478.
4. R.G.Halburd, R.J.Korhonen,"Meromorphic solutions of difference equations, integrability and the discrete Painleve equations", J. Phys. A., 40,(2007):1-38.
5. W.K.Hayman, "Meromorphic functions", Claredon Press,Oxford,(1964).
6. L.Kai, L.Xin-ling, C.Ting-bin, "Some results on

zeros and uniqueness of difference-differential polynomials", Appl. Math. J. Chinese Univ.,27:94-104, 2012.
7. I Lahiri, A Sarkar,"Uniqueness of a meromorphic function and its derivative",J. Inequal. Pure Appl. Math.,5: 2004.
8. *C. Sudhakar, B. Vasu, V.Ramachandra Prasad, N. Bhaskar Reddy,* Effect of thermophoresis Particle Deposition ...; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 1 (2013), Pg 51-53
9. Lipei Liu, "Uniqueness of meromorphic functions and differential polynomials", Comp. Math. Appl., 56:3236-3245, 2008.
10. K.Liu, L.Z.Yang, "Value distribution of the difference operator", Arch.Math., 92:270-278, 2009.
11. L.Yang, "Value Distribution Theory", Springer-Verlag Berlin,1993.
12. *D N Punith Kumar, indira R Rao, Dinesh P A,* Mathematical Modeling of Convective Diffusive Mass Transfer With ; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 241-244
13. C.C.Yang, X.H.Hua, "Uniqueness and value sharing of meromorphic functions", Ann. Acad. Sci. Fenn. Math.,22:395-406, 1997.
14. C.C.Yang, H.X.Yi, "Uniqueness Theory of Meromorphic Functions", Kluwer Academic Publishers, Dordrecht,2003; Chinese original: Science Press, Beijing, 1995.
15. X.Y.Zhang, J.F.Chen, W.C.Lin, "Entire or meromorphic functions sharing one value", Comp. Math. Appl.,56:1876-1883, 2008.
16. Jatinderdeep Kaur, integrability and *L1*-Convergence of Mixed; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 96-98
17. T.Zhang, W.Lu, "Uniqueness theorems on meromorphic functions sharing one value", Comp. Math. Appl., 55:2981-2992, 2008.

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