RESTRAINED TRIPLE CONNECTED DOMINATION NUMBER AND CONNECTIVITY OF GRAPHS

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Abstract: A subset D of V of a nontrivial graph G is called a dominating set of G if every vertex in V – D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. A dominating set is said to be restrained dominating set if every vertex in V – D is adjacent to at least one vertex in D as well as another vertex in V - D. A subset D of V of a graph G is said to be a restrained triple connected dominating set if D is a restrained dominating set and the induced subgraph <D> is triple connected. The minimum cardinality of a triple connected dominating set is called the restrained triple connected domination number of G and is denoted by $\gamma_{rtc}(G)$. The connectivity of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper we find an upper bound for the sum of the restrained triple connected domination number and connectivity of graphs and characterize the corresponding extremal graphs.

Keywords: Dominating set, restrained triple connected dominating set, connectivity.

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Introduction: By a graph $G(V, E)$ we mean a finite, simple, connected and undirected graph. The order and size of G are denoted by p and q respectively. The degree of any vertex v in G is the number of edges incident with v and is denoted by $d(v)$. The minimum and the maximum degree of G are defined as $\delta(G) = \min\{d(v)|v \in G\}$ and $\Delta(G) = \max\{d(v)|v \in G\}$. For graph theoretic terminology we refer Harary[1] and Haynes et.al [2].
A subset D of V of a nontrivial graph G is called a dominating set of G if every vertex in V – D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A graph G is said to be triple connected if any three vertices lie on a path in G. A dominating set D is said to be restrained dominating set if every vertex in V – D is adjacent to at least one vertex in D as well as another vertex in V - D. The minimum cardinality of a restrained dominating set is called the restrained domination number and is denoted by $\gamma_r(G)$. The restrained dominating set D is said to be a restrained triple connected dominating set (RTCD-set), if the induced sub graph <D> is triple connected. The minimum cardinality of a restrained triple connected dominating set is called the restrained triple connected domination number and is denoted by γ_{rtc} (G). The Connectivity of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. $H(m_1,m_2,\ldots,m_p)$ denotes the graph obtained from the graph H by attaching m_i pendant edges to the vertex $v_i \in V(H), 1 \le i \le p$.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J.Paulraj Joseph and S.Arumugam [6] proved that $\gamma(G)+\kappa(G) \le p$ and characterized the corresponding extremal graphs. In this paper, we obtain an upper bound for the sum of the restrained triple connected domination number and connectivity of a graph and characterize the corresponding extremal graphs. We use the following theorems.

Theorem 1.1. [4] For any graph G, $3 \le \gamma_{rtc}(G) \le p$ and $\gamma_{rtc}(G) \ne p-1$

Theorem 1.2. [1] For a graph G, $\kappa(G) \le \delta(G)$

Main Results:

Theorem 2.1 For any connected graph G, $\gamma_{rtc}(G) + \kappa(G) \le 2p-1$ and the equality holds if and only if G is isomorphic to either K_3 or K_4.

Proof: $\gamma_{rtc}(G) + \kappa(G) \le p + p - 1 = 2p - 1$. Let $\gamma_{rtc}(G) + \kappa(G) = 2p - 1$ then the only possible case is $\gamma_{rtc}(G) = p$ and $\kappa(G) = p - 1$. Since $\kappa(G) = p - 1$, G is a complete graph K_p. But $\gamma_{rtc}(K_p) = 3, p \ne 4$ and $\gamma_{rtc}(K_4) = 4$. This gives G is isomorphic to either K_3 or K_4. The converse is obvious.

Theorem 2.2 For any connected graph G, $\gamma_{rtc}(G) + \kappa(G) = 2p - 2$ if and only if G is isomorphic to $K_4 - e$ or P_3 or C_4.

Proof: Let $\gamma_{rtc}(G) + \kappa(G) = 2p - 2$ then the only possible case is $\gamma_{rtc}(G) = p$ and $\kappa(G) = p -$

2. Then $p-2 \leq \delta(G)$. If $\delta = p-1$ then G is isomorphic to a complete graph which is a contradiction. Hence $\delta = p-2$. Then G is isomorphic to $K_p - M$, where M is a matching in K_p. Hence $\gamma_{rtc}(G) = 3$ or 4. If $\gamma_{rtc}(G) = 3$ then G is isomorphic to P_3. If $\gamma_{rtc}(G) = 4$ then G is isomorphic to $K_4 - e$ or C_4. The converse is obvious.

Theorem 2.3 For any connected graph G, $\gamma_{rtc}(G) + \kappa(G) = 2p-3$ if and only if G is isomorphic to K_5 or P_4or $C_3(1,0,0)$.

Proof: Let $\gamma_{rtc}(G) + \kappa(G) = 2p-3$ then the possible cases are, (i) $\gamma_{rtc}(G) = p$ and $\kappa(G) = p-3$ and (ii) $\gamma_{rtc}(G) = p-2$ and$\kappa(G) = p-1$

Case 1. $\gamma_{rtc}(G) = p$ and$\kappa(G) = p-3$

Then $p-3 \leq \delta(G)$.If $\delta = p-1$ then G is isomorphic to a complete graph which is a contradiction. Hence $\delta = p-2$. Then G is isomorphic to $K_p - M$, where M is a matching in K_p.Hence$\gamma_{rtc}(G) = 3$ or 4. Thus p= 3 or 4. Since $(G) = p-3$, p=3 is not possible, hence p=4. But $\kappa(K_4 - e) = \kappa(C_4) = 4 \neq p-3$. Hence $\delta = p-3$. Let $X = \{v_1, v_2, \dots \dots . v_{p-3}\}$ be the vertex cut of G and let $X = \{x_1, x_2, x_3\}$.

Subcase 1.1: $\langle V - X\rangle = \overline{K_3}$

Then each vertex of V-X is adjacent to all the vertices in X. Then $\{x_1, x_2, v_1\}$ is a RTCD-set of G and hence $\gamma_{rtc}(G) = 3$. Thus $p = 3$ which is a contradiction.

Sub case 1.2: $\langle V - X\rangle = K_1 \cup K_2$

Let $x_1x_2 \in E(G)$ then x_3 is adjacent to all the vertices of X and x_1 , x_2 are not adjacent to at most one vertex in X. If $|X| \geq 2$ and $\langle X\rangle$ contains an isolated vertex then $\kappa(G) \leq 3$, hence $p \leq 6$. For these graphs $\gamma_{rtc}(G) \neq p$ which is a contradiction. Hence $|X| = 1$ then G is isomorphic to either P_4or $C_3(1,0,0)$. If $\langle X\rangle$ does not contain any isolated vertex and $|X| \geq 3$ then $\{x_1, x_2, x_3, v_1\}$where $v_1 \in N(x_1) \cup N(x_2)$ is a RTCD -set of G. Hence $\gamma_{rtc}(G) \leq 4$. Then p=4 which is a contradiction. Thus$|X| \geq 3$. If $|X| = 2$ then either $\kappa(G) \neq p-3$ or $\gamma_{rtc}(G) \neq p$ which is a contradiction.

Case 2. $\gamma_{rtc}(G) = p-2$ and$\kappa(G) = p-1$

If $\kappa(G) = p-1$ then G is a complete graph. But $\gamma_{rtc}(K_p) = 3, p \neq 4$. Hence$p = 5$. Thus G is isomorphic to K_5. The converse is obvious.

Theorem 2.4 For any connected graph G, $\gamma_{rtc}(G) + \kappa(G) = 2p-4$ if and only if G is isomorphic to P_5, or K_6or $K_5 - M$ where M is a matching in K_5.

Proof: Let $\gamma_{rtc}(G) + \kappa(G) = 2p-4$ then the possible cases are, (i) $\gamma_{rtc}(G) = p$ and $\kappa(G) = p-4$, (ii) $\gamma_{rtc}(G) = p-2$ and$\kappa(G) = p-2$ (iii) $\gamma_{rtc}(G) = p-3$ and$\kappa(G) = p-1$

Case 1. $\gamma_{rtc}(G) = p$ and$\kappa(G) = p-4$

Then $p-4 \leq \delta(G)$. If $\delta(G) = p-1$ then G is isomorphic to a complete graph which is a contradiction. If $\delta(G) = p-2$ then G is isomorphic to $K_p - M$, where M is the matching in K_p. Then $\gamma_{rtc}(G) = 3$ or 4. Hence p =3 or 4 which is contradiction. Suppose $\delta(G) = p-3$, let $X = \{v_1, v_2, \dots, v_{n-4}\}$ be the vertex cut of G and let $V - X = \{x_1, x_2, x_3, x_4\}$. Then it is clear that $\langle V - X\rangle$=$K_2 \cup K_2$. Hence each vertex of V-X is adjacent to all the vertices of X. Let $x_1x_2, x_3x_4 \in E(G)$. Then $\{x_1, x_3, v_1\}$ is a RTCD - set of G. Hence p ≤ 3which is a contradiction. Thus $\delta(G) = p-4$.

Case 1.1: $\langle V - X\rangle = \overline{K_4}$

Then every vertex of V-X is adjacent to all the vertices of X. If $|X| = 1$ then G is a star which is a contradiction. If $|X| \geq 2$ then $\{x_1, x_2, v_1\}$ is a RTCD- set of G. Hence p≤3 which is a contradiction.

Case 1.2: $\langle V - X\rangle$=$K_2 \cup K_2$

Let $x_1x_2, x_3x_4 \in E(G)$. Since $\delta(G) = p-4$ each x_i is not adjacent to atmost one vertex in X and each v_i is adjacent to atleast one vertex in V-X. If $|X| \geq 5$ then $\langle X\rangle$ does not contain any isolated vertex. Hence $\{x_1, x_3, v_1\}$ is a RTCD- set of G. Thus p≤3 which is a contradiction. Suppose $|X| = 1$ then X = $\{v_1\}$. If v_1 is adjacent to all the vertices in V-X then $\{x_1, x_2, v_1\}$ is a RTCD- set of G. Hence p≤3 which is a contradiction. If v_1 is not adjacent to exactly one vertex in V-X say x_4, then $\{x_4, x_3, v_1\}$ is a RTCD set of G. Hence p≤3 which is a contradiction. If v_1 is not adjacent to exactly two vertices in V-X say x_1 and $x_4$then G is isomorphic to P_5 and it is clear that $\gamma_{rtc}(P_5)$= 5 = p and $\kappa(P_5)$=1=p-4. Suppose $|X| = 2$ or 3 then $\{v_1, v_2, x_1, x_2\}$ or $\{v_1, v_2, x_3, x_4\}$ is a RTCD-set of G which is a contradiction to $\gamma_{rtc}(G) = p$

Sub case 1.3:$\langle V - X\rangle$=$K_2 \cup \overline{K_2}$

Let $x_1x_2 \in E(G)$and $x_3x_4 \in E(\bar{G})$. Then each x_i, i=1 or 2 is not adjacent to atmost one vertex in X and each x_j, j= 3 or 4 is adjacent to all the vertices in X. Then $\{x_1, x_3, v_1\}$ where $x_1 \in N(x_1)$ is a RTCD- set of G and hence p≤3 which is a contradiction.

Subcase 1.4:$\langle V - X\rangle$=$P_3 \cup K_1$

Let x_1 be the isolated vertex in $\langle V-X\rangle$and (x_2,x_3,x_4) be a path. Then x_1 is adjacent to all the vertices in X and x_2,v_4 are not adjacent to atmost one vertex in X. If x_2 is adjacent to all the vertices in X then $\{v_1,x_2,x_3\}$ is a RTCD- set of G and hence p≤3 which is a contradiction. Hence x_2,x_4 are not adjacent to exactly one vertex in X. Suppose x_2 is not adjacent to v_1 and x_4 is not adjacent to v_2 then $\{v_1,v_2,x_2,x_3,x_4\}$ is a RTCD-set of G and hence p=5 which is a contradiction. Suppose x_2 and x_4 are not adjacent to v_1. If v_1 is not adjacent to x_3 then v_1 should adjacent with all the vertices in X. hence $\{v_2,x_1,x_2\}$ is a RTCD-set of G which is a contradiction. If $v_1$is adjacent to x_3 then $\{v_1,x_2,x_3\}$ is a RTCD-set of G which is a contradiction.

Case 1.5:$\langle V-X\rangle$=$K_3\cup K_1$

Let x_1 be the isolated vertex in $\langle V-X\rangle$and $<\{x_2,x_3,x_4\}>$ be the complete graph. Then x_1 is adjacent to all the vertices in X and $\{x_2,x_3,x_4\}$ are not adjacent to atmost two vertices in X. Then in all different cases $\{v_1,x_1,x_2\}$ is a RTCD-set of G which is a contradiction.

Case 2.$\gamma_{rtc}(G)=p-2$ and$\kappa(G)=p-2$

If $\delta(G)=p-1$ then G is a complete graph which is a contradiction to $\kappa(G)=p-2$. Hence $\delta(G)=p-2$. Thus G is isomorphic to K_p-M, where M is a matching in K_p. Thus $\gamma_{rtc}(G)=3\ or\ 4$. If$\gamma_{rtc}(G)=3$ then $p=5$. Thus G is isomorphic to K_5-M , where M is a matching in K_5. If $\gamma_{rtc}(G)=4$ then $p=6$ which is a contradiction to $\gamma_{rtc}(K_6-M\ )$ = 3.

Case 3.$\gamma_{rtc}(G)=p-3$ and$\kappa(G)=p-1$

If $\kappa(G)=p-1$ then G is isomorphic to a complete graphK_p. Since $\gamma_{rtc}(K_p)=3$ if p≠4, we have $p=6$. Hence G is isomorphic to K_6. The converse is obvious.

Conclusion: In this paper we found an upper bound for the sum of restrained triple connected domination number and connectivity of graphs and characterized the corresponding extremal graphs up to 2p-4. The authors also obtained a similar characterization of a huge classes of graphs equals to 2p-5 and 2p-6, which will be reported in the subsequent papers.

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