ANOTHER PROOF OF MOHANTA'S FIXED POINT THEOREM IN G-METRIC SPACE

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Abstract: A nice proof of a recent result of Sushanta Kumar Mohanta (2012) is presentedby employing the well-known infimum property of real numbers without an appeal to the usual iterative procedure. An additional interesting consequence is that the obtained fixed point is shown as a G -contractive fixed point for f in the sense that the orbit $x, fx, \ldots, f^n x, \ldots$ at each $x \in X$ is G -convergent with limit p .

Keywords: Fixed Point,G-Cauchy Sequence, G-contractive Fixed Point, G-Metric Space, Self-map ,The Infimum Property.

Introduction: Let X be a nonempty set and $G: X \times X \times X \to \mathbb{R}$ such that

(G1) $G(x,y,z) \geq 0$ for all $x,y,z \in X$ with $G(x,y,z) = 0$ if $x = y = z$,

(G2) $G(x,x,y) > 0$ for all $x,y \in X$ with $x \neq y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,

(G4) $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(z,x,y) = G(y,z,x) = G(z,y,x)$ for all $x,y,z \in X$

(G5) $G(x,y,z) \leq G(x,w,w) + G(w,y,z)$ for all $x,y,z,w \in X$

Then the pair (X,G) is called a G -metric space with G -metric G on X .

Axioms (G4) and (G5) are referred to as the *symmetry* and the *rectangle inequality*(of G) respectively. The notion of G -metric space was introduced by Mustafa and Sims [15] in 2006.

Given a G -metric space (X,G) , define

$$\rho_G(x,y) = G(x,y,y) + G(x,x,y) \tag{1.1}$$

for all $x,y,z \in X$.

Then it is seen in [15] that ρ_G is a metric on X , and that the family of all G -balls $\{B_G(x,r): x \in X, r > 0\}$ is the base topology, called the G -metric topology $\tau(G)$ on X , where $B_G(x,r) = \{y \in X : G(x,y,y) < r\}$.

Further, it was shown that the G -metric topology coincides with the metric topology induced by the metric ρ_G , which allows us to readily transform many concepts from metric spaces into the setting of G -metric space.

Definition 1.1. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G -metric space (X,G) is said to be G -convergent with limit $p \in X$ if it converges to p in the G -metric topology $\tau(G)$.

It is also known from [15] that $G(x,y,z)$ is jointly continuous in all the three variables x,yand z .

Definition 1.2. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G -metric space (X,G) is said to be G -Cauchy if for every $\epsilon > 0$ there is a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $l,m,n \geq N$.

By Corollary 1 of Proposition 9 of [15], it follows that every G -convergent sequence in a G -metric space (X,G) is G -Cauchy.

Definition 1.3. A G -metric space (X,G) is said to be G -complete (or complete) if every G -Cauchy sequence in X converges in it.

For more examples and proofs of elementary results on G -metric space regarding the continuity and the convergence, one can refer to [2] and [15].

An extensive research has been done in recent years in G -metric spaces. To mention a few, we have Abbas and Rhoades [1], Aydi et al [3], Choudary et al [4], Jleli&Samet [6], Karapinar& Paul [7], Meenakshi et al [8], Mahanta&Mohanta [10], Mustafa [11], Mustafa et al [14], Shatanawi [17] and others cited in references. Several fixed point theorems established in the literature of G -metric space have followed the traditional iterative procedure to obtain a fixed point. For the first time the authors in 2013 initiated an alternative to this practice in the setting of G -metric space. In fact, the authors [16] employed the well-known infimum property of real numbers (See the next section) to prove the Banach contraction mapping theorem in G -metric space without an appeal to the usual iterative procedure.

In this paper, an alternative proof of a recent result of Sushant Kumar Mohanta [9] is presented by employing the well-known infimum property of real numbers without an appeal to the usual iterative procedure.

Main Result: We begin with the infimum property of real numbers, as stated below:

Lemma 2.1. Let $S \subset \mathbb{R}$ be nonempty and bounded below. Then $\alpha = \inf S$ exists.

An immediate consequence of Lemma **2.1**is:

Lemma 2.2. Let α be the infimum of $S \subset \mathrm{IR}$. Then there exists a sequence $\langle p_n \rangle_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} x_n = \alpha.$

From this definition of G-metric space, it immediately follows that

If $x, y \in X$ are such that

$G(x,y,y) \le 2G(x,x,y)$ then $x = y$ (2.1)

and that $G(x,y,y) \le 2G(x,x,y)$ (2.2)

Lemma 2.3 (Mustafa and Sims, [15]). The following statements are equivalent in a G-metric space (X,G):

a) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is G-convergent with limit $p \in X$,

b) $\lim_{n\to\infty} G(x_n, x_n, p) = 0$,

c) $\lim_{n\to\infty} G(x_n, p, p) = 0$.

Definition 2.1 (The authors, [16]).A fixed point p of f on a G-metric space (X,G) is a G-contractive fixed point of it if the orbital sequence $O_f(x) = \{x, fx, ..., f^n x, ...\}$ at each $x \in X$ converges to p.

Example 2.1. Let $X = [0,\infty)$ with

$$G(x,y,z) = \begin{cases} 0 \text{ if } x = y = z \\ \max\{x,y,z\} \text{ otherwise.} \end{cases}$$

Define $$fx = \begin{cases} 0 \text{ if } 0 \le x < \frac{1}{2}, \\ qx \text{ otherwise} \end{cases}$$

for all $x \in X$, where $0 \le q < 1$. Then we see that 0 is the unique fixed point of f and for each $x \in X$, the f-orbit $O_f(x) = \{x, qx, q^2 x, ..., q^n x, ...\}$ converges to 0. That is, 0 is a G-contractive fixed point of *f*.

Recently, Sushanta Kumar Mohanta [9] established the following theorem:

Theorem 2.1.*Suppose that (X,G) is a complete G-metric space and f, a self-map on X satisfying the following G-contraction condition such that*

$$G(fx,fy,fz) \le aG(x,y,z) + bG(x,fx,fx) + cG(y,fy,fy) + dG(z,fz,fz) + e\max\{G(x,fy,fy), G(y,fx,fx), G(y,fz,fz), G(z,fy,fy), G(z,fx,fx), G(x,fz,fz)\}$$

for all $x,y,z \in X$, (2.3)

where $a,b,c,d,e \ge 0$ with $a+b+c+d+2e < 1$.

Then f will have a unique fixed point p and f is continuous at p.

Alternative proof of Theorem 2.1 :

Let $S = \{G(x,fx,fx) : x \in X\}$. Note that S is a nonempty set of nonnegative numbers which is bounded below. Hence by Lemma 2.1, it has the infimum, say $a \ge 0$.

We claim that $a = 0$.

If possible, we suppose that $a > 0$. Now from (2.3) with $y = fx$ and $z = fx$ and the rectangle inequality (G5), we have

$$\begin{aligned} G(fx,f^2x,f^2x) &\le aG(x,fx,fx) + bG(x,fx,fx) + cG(fx,f^2x,f^2x) + dG(fx,f^2x,f^2x) \\ &+ e\max\{G(x,f^2x,f^2x), G(fx,fx,fx), G(fx,f^2x,f^2x), G(fx,f^2x,f^2x), G(fx,fx,fx), G(x,f^2x,f^2x),\} \\ &\le (a+b)G(x,fx,fx) + (c+d)G(fx,f^2x,f^2x) + e\max\{G(x,fx,fx), G(fx,f^2x,f^2x)\} \\ &\le (a+b)G(x,fx,fx) + (c+d)G(fx,f^2x,f^2x) + e[G(x,fx,fx) + G(fx,f^2x,f^2x)] \end{aligned}$$

Or $G(fx,f^2x,f^2x) \le \left(\dfrac{a+b+e}{1-(c+d+e)}\right) G(x,fx,fx).$ (2.4)

Since $\dfrac{a+b+e}{1-(c+d+e)} < 1$, from (2.4), it would follow that $G(fx,f^2x,f^2x) < a$ where $G(fx,f^2x,f^2x) \in S.$

In other words, a cannot be a lower bound of S, which is a contradiction.

Therefore, $a = \inf S = 0$.

By Lemma 2.2, we can choose the points $x_1, x_2, ..., x_n, ...$ in X such that

$G(x_n, fx_n, fx_n) \in S$ for $n = 1,2,3,...$

and $\lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0.$ (2.5)

We now prove that $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy. In fact, by repeated use of (G5) followed by (2.2), we get

$$\begin{aligned} G(x_n,x_m,x_m) &\le G(x_n,fx_n,fx_n) + G(fx_n,x_m,x_m) \\ &\le G(x_n,fx_n,fx_n) + [G(fx_n,fx_m,fx_m) + G(fx_m,x_m,x_m)] \\ &\le G(x_n,fx_n,fx_n) + [G(fx_n,fx_m,fx_m) + 2G(x_m,fx_m,fx_m)]. \end{aligned} \quad (2.6)$$

Now using (2.3) with $x = x_n, y = z = x_m$, (G5) and (2.2), it follows that

$$G(fx_n,fx_m,fx_m) \le aG(x_n,x_m,x_m) + bG(x_n,fx_n,fx_n) + cG(x_m,fx_m,fx_m) + dG(x_m,fx_m,fx_m) + e\max\{G(x_n,fx_m,fx_m), G(x_m,fx_m,fx_m) G(x_m,fx_m,fx_m), G(x_m,fx_m,fx_m),$$

$G(x_m, fx_n, fx_n) + G(x_n, fx_m, fx_m)\}$
$\leq aG(x_n, x_m, x_m) + bG(x_n, fx_n, fx_n)$
$+(c+d)G(x_m, fx_m, fx_m)$
$+e\max\{G(x_n, fx_m, fx_m), G(x_m, fx_n, fx_n),$
$G(x_m, fx_m, fx_m)\}$
$\leq aG(x_n, x_m, x_m) + bG(x_n, fx_n, fx_n)$
$+(c+d)G(x_m, fx_m, fx_m)$
$+e\max\{G(x_n, x_m, x_m) + G(x_m, fx_m, fx_m),$
$G(x_n, fx_n, fx_n) + 2G(x_n, x_m, x_m)\}$
$\leq aG(x_n, x_m, x_m) + bG(x_n, fx_n, fx_n)$
$+e[G(x_n, fx_n, fx_n) + G(x_m, fx_m, fx_m)$
$+2G(x_n, x_m, x_m)]$

Or

$$G(fx_n, fx_m, fx_m) \leq (a+2e)G(x_n, x_m, x_m) +(b+e)G(x_n, fx_n, fx_n) +(c+d+e)G(x_m, fx_m, fx_m). \quad (2.7)$$

Substituting (2.7) in (2.6), we get

$$G(x_n, x_m, x_m) \leq \left(\frac{1+b+e}{1-a-2e}\right)G(x_n, fx_n, fx_n) + \left(\frac{2+c+d+e}{1-a-2e}\right)G(x_m, fx_m, fx_m).$$

Employing the limit as $m, n \to \infty$ in this and using {2.5},we get

$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0,$ proving that $\langle x_n \rangle_{n=1}^{\infty}$ is G -Cauchy.

Since X is G -complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = p. \quad (2.8)$$

Again, by repeated application of rectangle inequality (G5), we have

$$G(p, fp, fp) \leq G(p, fx_n, fx_n) + G(fx_n, fp, fp) \leq [G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] +G(fx_n, fp, fp). \quad (2.9)$$

Now, (2.3) with $x = x_n, y = z = p$ implies

$G(fx_n, fp, fp) \leq aG(x_n, p, p) + bG(x_n, fx_n, fx_n)$
$+cG(p, fp, fp) + dG(p, fp, fp)$
$+e\max\{G(x_n, fp, fp), G(p, fx_n, fx_n),$
$G(p, fp, fp),\ G(p, fp, fp),$
$G(p, fx_n, fx_n), G(x_n, fp, fp)\}$
$\leq aG(x_n, p, p) + bG(x_n, fx_n, fx_n)$
$+(c+d)G(p, fp, fp)$
$+e\max\{G(x_n, fp, fp), G(p, fp, fp),$
$G(p, fx_n, fx_n)\}$
$\leq aG(x_n, p, p) + bG(x_n, fx_n, fx_n)$
$+(c+d)G(p, fp, fp)$
$+e\max\{G(p, fp, fp) + G(x_n, p, p),$
$G(p, x_n, x_n) + G(x_n, fx_n, fx_n)\}$
$\leq aG(x_n, p, p) + bG(x_n, fx_n, fx_n)$
$+(c+d)G(p, fp, fp)$
$+e[2G(x_n, p, p) + G(p, fp, fp) + G(x_n, fx_n, fx_n)]$

or

$$G(fx_n, fp, fp) \leq (a+2e)G(x_n, p, p) +(b+e)G(x_n, fx_n, fx_n) +(c+d+e)G(p, fp, fp). \quad (2.10)$$

Substituting (2.10) in (2.9), we get

$$G(p, fp, fp) \leq \left(\frac{2+a+2e}{1-c-d-e}\right)G(x_n, p, p) + \left(\frac{b+e}{1-c-d-e}\right)G(x_n, fx_n, fx_n). \quad (2.11)$$

Proceeding the limit as $n \to \infty$ and then using (2.5), (2.8) and Lemma 2.3, we get from (2.11),

$G(p, fp, fp) \leq \left(\frac{2+a+2e}{1-c-d-e}\right)(0) + \left(\frac{b+e}{1-c-d-e}\right)(0)$ so that $G(p, fp, fp) = 0$, which in view of (2.1) yields $fp = p$. That is p is a fixed point of f .

Uniqueness: Suppose q is another fixed point of f . That is $fq = q$.

Then from (2.3)with $x = p$ and $y = z = q$, we have

$G(p,q,q) = G(fp, fq, fq)$
$\leq aG(p,q,q) + bG(p, fp, fp)$
$+cG(q, fp, fp) + dG(q, fq, fq)$
$+e\max\{G(p, fq, fq), G(q, fp, fp),$
$G(q, fq, fq), G(p, fq, fq),$
$G(q, fp, fp), G(p, fq, fq),\}$
$\leq aG(p,q,q) + 0 + 0 + 0$
$+e\max\{G(p,q,q), G(q, p, p)\}$
$\leq aG(p,q,q)$
$+e\max\{G(p,q,q), 2G(p,q,q)\}$
$= aG(p,q,q) + 2eG(p,q,q)$

Or $(1-a-2e)G(p,q,q) \leq 0$ so that $G(p,q,q) = 0$ which together with (2.1) implies that $p = q$. That is p is unique fixed point of f .

The G-continuity of f at p follows from [9]. This completes the proof.

Remark 2.1. It is significant to establish that p is a G -contractive fixed point of f provided

$a+3b+3e<1$. Indeed taking $y=z=p$ in (2.3) and using (G5), we get $G(f^n x,p,p)=G(f^n x,fp,fp)$

$$\le aG(f^{n-1}x,p,p)+bG(f^{n-1}x,f^n x,f^n x)$$
$$+cG(p,fp,fp)+dG(p,fp,fp)$$
$$+e\max\{G(f^{n-1}x,fp,fp),G(p,f^n x,f^n x),$$
$$G(p,fp,fp),G(p,fp,fp),$$
$$G(p,f^n x,f^n x),G(f^{n-1}x,fp,fp)\}$$
$$\le aG(f^{n-1}x,p,p)+bG(f^{n-1}x,f^n x,f^n x)$$
$$+e\max\{G(f^{n-1}x,p,p),G(p,f^n x,f^n x)\}$$
$$\le aG(f^{n-1}x,p,p)+b[G(f^{n-1}x,p,p)$$
$$+G(p,f^n x,f^n x)]$$
$$+e\max\{G(f^{n-1}x,p,p),2G(f^n x,p,p)\}$$
$$\le (a+b)G(f^{n-1}x,p,p)+2bG(f^n x,p,p)$$
$$+e[G(f^{n-1}x,p,p)+2G(f^n x,p,p)]$$

or $G(f^n x,p,p)\le c\cdot G(f^{n-1}x,p,p)$.

where $c=\dfrac{a+b+e}{1-2b-2e}<1$. By induction, we have $G(f^n x,p,p)\le c^n G(fx,p,p)$, whichas $n\to\infty$ gives $f^n x\to p$ for each $x\in X$, in view of Lemma 2.3. Thus p is a G-contractive fixed point of f.

References:

1. Abbas, M., Rhoades, B. E., Common fixed point results for noncommuting mappings without continuity in generalized metric spaces, Appl. Math. and Comp. 21(5) (2009), 262-269
2. AnamikaGanguly, A study on fixed points of operators in different spaces, Thesis University of Burdwan, 2012
3. Aydi, H., Erdal, K., Sathanawi, W., Tripled Fixed Point Results in Generalized Metric spaces, Journal of Applied Mathematics(2012), Article ID 314279,1-9
4. BinayakChoudhary, S., Sanjay Kumar, Asha, Krishnapanda Das, Some Fixed Point Theorems in *G*-metric spaces, Math. Sci.Letters,1(1) (2012), 25-31
5. Chouhan, S. K., Fixed points of Contraction mappings on metric and G-metric spaces, Dissertation, Thapar University, 2013
6. Jleli, M., Samet, B., Remarks on *G*-metric spaces and fixed point theorems, Fixed Point Theory and Applications-2012, (2012): 210, 1-7
7. Karapinar, E., Ravi Paul, A., Further Remarks on *G*-metric spaces , Fixed point theory and Applications,154 (2013), 1-19, doi: 10.1186/1687-1812-2013-154
8. Meenakshi, G., Madhu, A., Renu, C., Common Fixed Point Results in *G* -metric spaces and Applications, International Journal of Computer Applications 43(11) (2012), 38-42
9. Mohanta, S. K., Fixed Point Theorems in *G* -metric spaces, An.St.Univ.Ovidius Constanta, 20(1) (2012), 285-306
10. Mohanta, S. K., Mohanta, S., A common Fixed Point Theorem *G*–metric spaces, CUBO A Mathematical Journal, 14(3) (2012), 85-101
11. Mustafa, Z., Some New Common Fixed Point Theorems under Strict Contractive Conditions in *G* -Metric Spaces , journal of Applied athematics (2012) ,Article ID 248397, 1-21.
12. Mustafa, Z., Obiedat, H., A fixed point theorem of Reich in G-Metric Spaces, CUBO A Math. Jour. 12(1) (2010), 83-93
13. Mustafa, Z., Obiedat, H., Awawdeh, F., Some fixed point theorem for mapping on complete G-Metric Spaces, Fixed Point Theory and Applications (2008), Article ID 189870, 1-12
14. Mustafa, Z., Shatanawi,W., Bataineh,M., Existence of Fixed Point Results in G-Metric Spaces , International journal Of Mathematics and Mathemaical sciences(2009),ArticleID283028, 1-10.
15. *K.Sireesha, Dr G. Ravindra Babu,* A Resource Allocation Model for Hospital Administration; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 1 (2013), Pg 107-110
16. Mustafa, Z., Sims, B., A new approach to generalized metric spaces, Journal of Nonlinear and Convex Anal. 7(2) (2006), 289-297
17. Phaneendra, T.,Kumara Swamy, K., Uniquefixed pointinGmetricspacethroughgreatestlowerboundproperties,NoviSadJ.Math.,43(2)(2013),107-115
18. Shatanawi,W., Fixed Point Theory for Contractive Mappings Satisfying ϕ - Maps in *G* -Metric Spaces, Fixed point theory and applications(2010), Article ID 181650, 1-9
19. Vats, R. K., Kumar, S., Sihag,V., Fixed Point Theorems in Complete *G*-metric space,Fasc. Math. 47 (2011), 127-138

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