PSEUDO-DEFICIENCY AND UNIQUENESS THEOREM IN ANNULI

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Abstract: In this paper, we discuss the influence of Pseudo-deficiency on the uniqueness problem of meromorphic functions on annuli, and we extend the Nevanlinna value distribution theory for meromorphic functions on annuli.

Keywords: Nevanlinna Theory; Pseudo-deficiency; the uniqueness of meromorphic functions on annuli.

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1.Introduction: The uniqueness theory of meromorphic functions in annuli is an interesting problem, and many researchers like Korhonen R [4], T.B.Cao, H.X.Yi, H.Y.Xu [1], and A.Ya.Khrystiyanya, A.A.Kondratyuk [2,3] and others have done a lot of work in this area. In this article, we discuss the uniqueness theory of meromorphic functions dealing with Pseudo-deficiency and extend some results from meromorphic functions to the meromorphic functions in annuli. For standard notations, definitions and some preliminary theorems refer [1, 2, 3].

2.Basic notations in the Nevanlinna Theory on annuli:

Let f be a meromorphic function on the annulus $A=\left\{z:\frac{1}{R_0}<|z|<R_0\right\}$. We recall classical notations of Nevanlinna theory as follows

$$N(R,f)=\int_0^R\frac{n(t,f)-n(0,f)}{t}dt+n(0,f)\log R,$$

$$m(R,f)=\int_0^{2\pi}\log^+|f(Re^{i\theta})|d\theta,$$

$$T(R,f)=N(R,f)+m(R,f),$$

Where $\log^+x=\max\{\log x,0\}$ and $n(t,f)$ is the counting function of poles of the function f in $\{z:|z|\le t\}$ Here we show the notations of the Nevanlinna theory on annuli. Let

$$N_1(R,f)=\int_{\frac{1}{R}}^1\frac{n_1(t,f)}{t}dt,$$

$$N_2(R,f)=\int_1^R\frac{n_2(t,f)}{t}dt,$$ and

$N_0(R,f)=N_1(R,f)+N_2(R,f)$, where $n_1(t,f)$ and $n_2(t,f)$ are the counting functions of the poles of the function f in $\{z:t<|z|\le 1\}$ and $\{z:1<|z|\le t\}$, respectively. The Nevanlinna characteristic of f on the annulus A is defined by

$$T_0(R,f)=m_0(R,f)+N_0(R,f)-2m(1,f).$$

We use $\overline{n}_1^{k)}\left(t,\frac{1}{f-a}\right)$ or $\overline{n}_1^{(k}\left(t,\frac{1}{f-a}\right)$ to denote the counting function of poles of the function $1/f-a$ with the multiplicities $\le k(or>k)$ in $\{z:t<|z|\le 1\}$, each point counted only once. Similarly, we can give the notations $\overline{N}_1^{k)}(t,f),\overline{N}_1^{(k}(t,f),\overline{N}_2^{k)}(t,f),\overline{N}_2^{(k}(t,f)$, $\overline{N}_0^{k)}(t,f),\overline{N}_0^{(k}(t,f)$.

Theorem 2.1.[1] (The Second Fundamental Theorem). Let $f(z)$ be a non-constant meromorphic function on the annulus $A(R_0)=\left\{z:\frac{1}{R_0}<|z|<R_0\right\}$, where $1<R_0<+\infty$. Let $a_1,a_2,\dots,a_q$ be q-distinct complex numbers in the extended complex plane $\overline{C}=C\cup\infty$, let $k_1,k_2,\dots,k_q$ be q positive integers, and let $\lambda\ge 0$. then

(i) $(q-2)T_0(R,f)$

$$<\sum_{j=1}^q N_0\left(R,\frac{1}{f-a_j}\right)-N_0^{(1)}(R,f)+S(R,f),$$

(ii) $(q-2)T_0(R,f)<\sum_{j=1}^q N_0\left(R,\frac{1}{f-a_j}\right)$
$+S(R,f)$

$(iii)(q-2)T_0(R,f)<\sum_{j=1}^q\frac{k_j}{k_j+1}\overline{N}_0^{k_j)}\left(R,\frac{1}{f-a_j}\right)$[

$+\sum_{j=1}^q\frac{1}{k_j+1}N_0\left(R,\frac{1}{f-a_j}\right)+S(R,f),$

$$(iv)\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T_0(R,f) < \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_0^{k_j)}\left(R,\frac{1}{f-a_j}\right)+S(R,f).$$

Where,

$$N_0^{(1)}(R,f)=N_0\left(R,\frac{1}{f'}\right)+2N_0(R,f)-N_0(R,f'),$$

and $S(R,f)$ satisfies the properties (i) and (ii) mentioned in Theorem 2.2 [3].

3.Pseudo-deficiency in Annulus: Let f be a meromorphic function on the $A(R_0)=\left\{z:\frac{1}{R_0}<|z|<R_0\right\}$, where $1<R_0<+\infty$.

Let a be anarbitrary complex number, and k be a positive integer. Corresponding to usual deficiency in annulus,we define the following Pseudo-deficiency in annulus as follows

$$\delta_0^{k)}(a,f)\le 1-\limsup_{r\to\infty}\frac{\overline{N}_0^{k)}(R,f-a)}{T_0(R,f)}.$$

The Pseudo-deficiency in the annulus has the following properties.

Theorem 3.1. Let $f(z)$ be a non-constant meromorphic function on the annulus $A(R_0)=\left\{z:\frac{1}{R_0}<|z|<R_0\right\}$, where $1<R_0<+\infty$ and k be a positive integer. Then

(i) For any complex number a, $0\le\delta_0^{k)}(a,f)\le 1$;

(ii) There exist at most countably many complex number with $\delta_0^{k)}(a,f)\ge 0$.

Moreover

$$\sum_a\delta_0^{k)}(a,f)\le\frac{2(k+1)}{k} \quad (3.1)$$

Proof: (i) follows immediately from the definition of Pseudo-deficiency on the annulus.

(ii) Now assume that $a_j\,(j=1,2,\dots,q)$ are $q>2$ distinct complex numbers. Then it follows from Theorem 2.1 that

$$\left(q-2-\frac{q}{k+1}\right)T_0(R,f) < \sum_{j=1}^{q}\frac{k}{k+1}\overline{N}_0^{k)}\left(R,\frac{1}{f-a_j}\right)+S(R,f),$$

$$q-\frac{q}{k+1}-\frac{k}{k+1}\sum_{j=1}^{q}\frac{\overline{N}_0^{k_j)}\left(R,\frac{1}{f-a_j}\right)}{T_0(R,f)} < 2+\frac{S(R,f)}{T_0(R,f)},$$

$$\frac{k}{k+1}\sum_{j=1}^{q}\left(1-\frac{\overline{N}_0^{k_j)}\left(R,\frac{1}{f-a_j}\right)}{T_0(R,f)}\right)<2+\frac{S(R,f)}{T_0(R,f)}.$$

Therefore,

$$\sum_{j=1}^{q}\delta_0^{k)}(a,f)$$

$$=\sum_{j=1}^{q}\liminf_{r\to\infty}\left(1-\limsup_{r\to\infty}\frac{\overline{N}_0^{k)}(R,f-a)}{T_0(R,f)}\right),$$

$$\le\liminf_{r\to\infty}\sum_{j=1}^{q}\left(1-\limsup_{r\to\infty}\frac{\overline{N}_0^{k)}(R,f-a)}{T_0(R,f)}\right),$$

$$\le\liminf_{r\to\infty}\frac{k+1}{k}\left(2+\frac{S(R,f)}{T_0(R,f)}\right),$$

$$=\frac{2(k+1)}{k} \quad (3.2)$$

Furthermore, since q is arbitrary, (3.1) follows immediately from (3.2). From (3.1), one can conclude easily that there can be only countably many complex number a with $\delta_0^{k)}(a,f)\ge 0$.

4. Uniqueness Theorems in Annulus

Let $E(a,f)=\{z\in A: f(z)-a=0\}$, where each zero with multiplicity m times. If we ignore multiplicity, then the set is denoted by $\overline{E}(a,f)$. We use $\overline{E}_{k)}(a,f)$ to denote the set of zeros of $f-a$ with multiplicities no greater than k, in which each zero is counted only once.

Theorem 4.1. Let $f(z)$ be a non-constant meromorphic function on the annulus $A(R_0)=\left\{z:\frac{1}{R_0}<|z|<R_0\right\}$, where $1<R_0<+\infty$.

Then $f(z)$ can be uniquely determined by $q(=5+[2/k])$ sets $\overline{E}_{k)}(a_j,f)(j=1,2,\dots,q)$ where $a_j\,(j=1,2,\dots,q)$ are q distinct numbers and $[2/k]$ denotes the largest integer less than or equal to $[2/k]$.

Proof : Let $f(z)$ be a non-constant meromorphic function on the annulus $A(R_0)=\left\{z:\frac{1}{R_0}<|z|<R_0\right\}$, where $1<R_0<+\infty$, satisfying

$$\overline{E}_{k)}(a_j,f)=\overline{E}_{k)}(a_j,g),\ (j=1,2,\ldots,q) \quad (4.1)$$

We may assume that without loss of generality that $a_j\,(j=1,2,\ldots,q)$ are all finite numbers, otherwise, a suitable linear transformation will be done. Assume $f(z)\neq g(z)$. Then from Theorem 2.1, we have

$$\left(q-2-\frac{q}{k+1}\right)T_0(R,f)$$
$$<\sum_{j=1}^{q}\frac{k}{k+1}\overline{N}_0^{k)}\left(R,\frac{1}{f-a_j}\right)+S(R,f). \quad (4.2)$$

It follows from equation (4.1) that

$$\sum_{j=1}^{q}\overline{N}_0^{k)}\left(R,\frac{1}{f-a_j}\right)\leq N_0\left(R,\frac{1}{f-g}\right)$$
$$\leq T_0(R,f)+T_0(R,g)+O(1).$$

On the other hand, (4.2) give

$$\left(q-2-\frac{q}{k+1}\right)T_0(R,f)$$
$$<\frac{k}{k+1}T_0(R,f)+\frac{k}{k+1}T_0(R,g)+S(R,f). \quad (4.3)$$

Similarly,

$$\left(q-2-\frac{q}{k+1}\right)T_0(R,g)$$
$$<\frac{k}{k+1}T_0(R,f)+\frac{k}{k+1}T_0(R,g)+S(R,f). \quad (4.4)$$

Combining (4.3) and (4.4), we obtain

$$\left(q-2-\frac{q}{k+1}\right)[T_0(R,f)+T_0(R,g)]$$
$$<\frac{2k}{k+1}T_0(R,f)+\frac{2k}{k+1}T_0(R,g)$$
$$+S(R,f)+S(R,g),$$
$$\left(q-2-\frac{q}{k+1}-\frac{2k}{k+1}\right)[T_0(R,f)+T_0(R,g)]$$
$$<S(R,f)+S(R,g),$$
$$(kq-4k-2)[T_0(R,f)+T_0(R,g)]$$
$$<S(R,f)+S(R,g). \quad (4.5)$$

Note that $q=5+[2/k]$ and thus $kq-4k-2>0$. Therefore, (4.5) does not hold.

This is a contradiction and hence $f(z)\equiv g(z)$.

Corollary4.1:Let $f(z)$ be a non-constant meromorphic function on the annulus $A(R_0)=\left\{z:\frac{1}{R_0}<|z|<R_0\right\}$, where $1<R_0<+\infty$. Then $f(z)$ can be uniquely determined by seven points sets : $\overline{E}_{k)}(a_j,f)(j=1,2,\ldots,7)$ or six points sets: $\overline{E}_{k)}(a_j,f)(j=1,2,\ldots,6)$ or five points sets: $\overline{E}_{k)}(a_j,f)(j=1,2,\ldots,5)$.

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