ON RELATIVE p-BOUNDS OF LINEAR OPERATORS

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Abstract: A closed operator A that is not bounded may be bounded with respect to anotherunbounded closed operator T. In this situation, A is said to be bounded relative to T. Relative bounds are defined for relatively bounded operators and these bounds are used to define certain relative approximation numbers of unbounded operators. In this paper we extend the notion of relative boundedness to higher orders and higherorder relativebounds are introduced. Using these generalized relative bounds,generalized relative approximation numbers are defined andcertain properties of these numbers are discussed in this paper.

Key words: generalized bounds, relative approximation numbers, relative boundedness,

1. Relative p-bounds:

Definition 1.1 Let Xbe a Banach space. Let T and A be densely-defined linear operators with domain spaces $D(T)$ and $D(A)$ respectively contained in X such that $D(T) \subset D(A)$. Suppose there are non-negative real numbers a and b with

$\|Au\| \leq a\|u\| + b\|Tu\|$, for every $u \in D(T)$. ...(1.1)

Then$A$is said to be relatively bounded with respect to T or T-bounded.

The greatest lower bound b_0 of all possible b in (1.1) is called the relative bound of A with respect to T or the T-bound of A. We may denote the T-bound of$A$ by $b_T(A)$and also we denote the class of T-bounded operator by $\mathcal{B}_T(X)$.

Note. Every bounded operator A is T-bounded with$b_T(A) = 0$. Thus,
$\mathcal{B}(X) \subset \mathcal{B}_T(X)$ for any T, where $\mathcal{B}(X)$ denotes the class of bounded operators on$X$.

Proposition 1.2 *Let$p$be an integer≥ 1. $A$is$T$-bounded if and only if there are non-negative reals m and $n$such that*

$\|Au\|^p \leq m^p\|u\|^p + n^p\|Tu\|^p, \quad$ *for every* $u \in D(T)$. ...(1.2)

Proof. Suppose (1.2) holds. Then for$u \in D(T)$,

$\|Au\|^p \leq m^p\|u\|^p + n^p\|Tu\|^p, u \in D(T)$

$\leq m^p\|u\|^p + pC_1m^{p-1}n\|u\|^{p-1}\|Tu\| + pC_2m^{p-2}n^2\|u\|^{p-2}\|Tu\|^2 + \cdots + n^p\|Tu\|^p$

$= (m\|u\| + n\|Tu\|)^p$.

$\therefore \|Au\|^p \leq (m\|u\| + n\|Tu\|)^p$.

Thus, $\|Au\| \leq m\|u\| + n\|Tu\|$ for every $u \in D(T)$.

Hence $A$is T-bounded.

To prove the converse part, assume that$A$ is T-bounded.

Then, for $u \in D(T)$, $\|Au\| \leq a\|u\| + b\|Tu\|$.

$\|Au\|^p \leq (a\|u\| + b\|Tu\|)^p$

$= a^p\|u\|^p + pC_1a^{p-1}b\|u\|^{p-1}\|Tu\| + pC_2a^{p-2}b^2\|u\|^{p-2}\|Tu\|^2 + \cdots + b^p\|Tu\|^p$.

$\|Au\|^p \leq a^p\|u\|^p + pC_1a^{p-1}b\|u\|^{p-1}\|Tu\| + pC_2a^{p-2}b^2\|u\|^{p-2}\|Tu\|^2 + \cdots + b^p\|Tu\|^p,$

$u \in D(T)$ ... (1.3)

Let $E = \{u \in D(T): \|u\| \leqq \|Tu\|\}$

and $F = \{u \in D(T): \|u\| > \|Tu\|\}$.

Then $D(T) = E \cup F$.

For $u \in E$ (1.3) implies

$$\|Au\|^p \leq a^p\|u\|^p + pC_1a^{p-1}b\|Tu\|^{p-1}\|Tu\| + pC_2a^{p-2}b^2\|Tu\|^{p-2}\|Tu\|^2 + \cdots + b^p\|Tu\|^p$$
$$= a^p\|u\|^p + (pC_1a^{p-1}b + pC_2a^{p-2}b^2 + \cdots + b^p)\|Tu\|^p. \quad \text{... (1.4)}$$

For$u \in F$ (1.3) implies

$$\|Au\|^p \leq a^p\|u\|^p + pC_1a^{p-1}b\|u\|^{p-1}\|u\| + pC_2a^{p-2}b^2\|u\|^{p-2}\|u\|^2 + \cdots + b^p\|Tu\|^p$$
$$= (a^p + pC_1a^{p-1}b + \cdots + pC_{P-1}ab^{p-1})\|u\|^p + b^p\|Tu\|^p. \quad \text{... (1.5)}$$

Thus, $\|Au\|^p \leq (a^p + pC_1a^{p-1}b + \cdots + pC_{P-1}ab^{p-1})\|u\|^p +$

$(pC_1a^{p-1}b + pC_2a^{p-2}b^2 + \cdots + b^p)\|Tu\|^p, \quad u \in D(T)$

$= m^p\|u\|^p + n^p\|Tu\|^p, \quad u \in D(T)$,

where $m^p = (a^p + pC_1a^{p-1}b + \cdots + pC_{P-1}ab^{p-1})andn^p = (pC_1a^{p-1}b + pC_2a^{p-2}b^2 + \cdots + b^p)$.

Definition 1.3 Let X be a Banachspace. Let $T$and Abe densely-defined linear operators with domain spaces $D(T)$ and $D(A)$ respectively contained in X such that $D(T) \subset D(A)$. Suppose there are non-negative real numbers m and n with

$$\|Au\|^p \leq m^p\|u\|^p + n^p\|Tu\|^p, \text{ for every } u \in D(T). \quad \text{... (1.6)}$$

Then A is said to be relatively p-bounded with respect to T or $T(p)$-bounded.
The greatest lower bound of all possible $n$satisfying (1.6) is called the relative p-bound of $A$with respect to T, or simply, the $T(p)$-bound of $A$and is denoted by$b_{T(p)}(A)$.Then (1.6) implies,

$$\|Au\|^p \le m^p\|u\|^p + \left(b_{T(p)}(A)\right)^p\|Tu\|^p, u \in D(T). \quad \text{... (1.7)}$$

The infimum of all possible m satisfying (1.7) if it exists, is called the $T(p)$-cobound of$A$, and is denoted by $b'_{T(p)}(A)$.

$$\|Au\|^p \le \left(b'_{T(p)}(A)\right)^p \|u\|^p + \left(b_{T(p)}(A)\right)^p\|Tu\|^p,\ u\in \mathrm{D}(T).$$

If there is no m satisfying (1.7), we set $b'_{T(p)}(A) = \infty$.

Remark. For a bounded operator A,$b_{T(p)}(A) = 0$ and$b'_{T(p)}(A) = \|A\|$, for any T.

Proof. Let $A \in B(X)$and Tbe any densely defined operator in X.
Then, for$u \in \mathrm{D}(T)$, $\|Au\| \le \|A\|\|u\|$.
So,$\|Au\|^p \le \|A\|^p\|u\|^p \le \|A\|^p\|u\|^p + 0\|Tu\|^p$.
Hence, $b_{T(p)}(A) = 0$.

$$\text{Now,}b'_{T(p)}(A) = \inf\left\{m \ge 0/\|Au\|^p \le m^p\|u\|^p + \left(b_{T(p)}(A)\right)^p \|Tu\|^p \text{ for all}u \in D(T)\right\}$$
$$= \inf\{m \ge 0/\|Au\|^p \le m^p\|u\|^p\text{for all}u \in D(T)\}$$
$$= \inf\{m \ge 0/\|Au\|^p \le m^p\|u\|^p\text{for all}u \in X\},$$

$A$is bounded and $D(T)$ is dense in$X$
$= \|A\|$.

2. Relative p-approximation numbers

Definition 2.1 Let$T$be a closed operator in X and A be T-bounded. For $k = 1,2,3,\ldots$the k^{th}relative p-approximation number $\varphi^{(p)}_{k,T}(A)$ is defined as

$$\varphi^{(p)}_{k,T}(A) = g.l.b\left\{b'_{T(p)}(A-F)/F \in \mathcal{B}(X), rankF \le k-1\right\} \quad \text{...(2.1)}$$

Also we define $\varphi^{(p)}_{\infty,T}(A) = g.l.b\left\{b'_{T(p)}(A-K)/K \in \mathcal{K}(X)\right\}$, ...(2.2)
where$\mathcal{K}(X)$ denote the class of compact operators on X.
We note that $\varphi^{(p)}_{k,T}(A)$ can be ∞.
It is clear from the definition that

$$b'_{T(p)}(A) = \varphi^{(p)}_{1,T}(A) \ge \varphi^{(p)}_{2,T}(A) \ge \cdots \varphi^{(p)}_{\infty,T}(A) \ge 0\,. \quad \text{... (2.3)}$$

Proposition 2.2 *If A is a bounded operator,$\varphi^{(p)}_{k,T}(A) = a_k(A)$, where a'_ks are the classical approximation numbers defined by*
$a_k(A) = g.l.b\{\|A-F\|/F \in \mathcal{B}(X),\ rankF \le k-1\}$.

Proof. $\varphi^{(p)}_{k,T}(A) = g.l.b\left\{b'_{T(p)}(A-F)/F \in \mathcal{B}(X),\ rankF \le k-1\right\}$
and since$A$ is a bounded operator, $b'_{T(p)}(A-F) = \|A-F\|$, by the remark after Definition 1.3.
so, $\varphi^{(p)}_{k,T}(A) = g.l.b\left\{b'_{T(p)}(A-F)/F \in \mathcal{B}(X),\ rankF \le k-1\right\} = a_k(A)$.

Remark. From proposition 2.2, we observe that the approximation numbers $\varphi^{(p)}_{k,T}(A)$ are generalizations of the numbers $a_k(T)$ to a class of unbounded operators.

3. Generalized p-bounds

Proposition3. 1. *$A$is$T$–bounded if and only if there exists $\eta < \infty$ such that*

$$\|Au\|^p \leqq \eta^p(\|u\|^p + \|Tu\|^p)\,, \qquad u\in \mathrm{D}(T).$$

Proof: Let$A$be T-bounded.
By proposition 1.2, there are constants$m$and $n$such that
$\|Au\|^p \le m^p\|u\|^p + n^p\|Tu\|^p,\ u\in \mathrm{D}(T)$.

$$\text{Let } \eta^p = max\{m^p, n^p\}.$$

Then, $\|Au\|^p \le \eta^p\|u\|^p + \eta^p\|Tu\|^p,\ u\in \mathrm{D}(T)$
$= \eta^p(\|u\|^p + \|Tu\|^p),\ u\in \mathrm{D}(T)$.

$$\|Au\|^p \le \eta^p(\|u\|^p + \|Tu\|^p), \qquad u\in \mathrm{D}(T).$$

Conversely, if $\|Au\|^p \le \eta^p(\|u\|^p + \|Tu\|^p)$, for all $u\in \mathrm{D}(T)$,
then $\|Au\|^p \le \eta^p\|u\|^p + \eta^p\|Tu\|^p,\ u\in \mathrm{D}(T)$.
Hence, comparision with proposition 1.2 with $c = d = \eta$, we see that A is T-bounded.

Definition 3.2. Let Abe $T(p) -$ bounded in X. The infimum of all'η'satisfying

$$\|Au\|^p \le \eta^p(\|u\|^p + \|Tu\|^p), \qquad u\in \mathrm{D}(T) \quad \text{... (3.1)}$$

is called the generalized relative$p -$bound of A with respect to T, or simply thegeneralized $T(p)-$bound of A and is denoted by$g_{T(p)}(\mathrm{A})$.

From (3.1), we see that$\|Au\|^p \le \left(g_{T(p)}(A)\right)^p (\|u\|^p + \|Tu\|^p)$, $u \in \mathrm{D}(T)$. ...(3.2)

Remark.

1. $g_{T(p)}(A) = 0$ if and only if $\|Au\|^p \le 0\,(\|u\|^p + \|Tu\|^p)$ for all $u \in \mathrm{D}(T)$.
   if and only if $\|Au\|^p \le 0$ for all $u \in \mathrm{D}(T)$
   if and only if $\|Au\| \le 0$ for all $u \in \mathrm{D}(T)$
   if and only if$A = 0$.
2. $g_{T(p)}(\lambda A) = |\lambda| g_{T(p)}(A) foranyscalar\lambda$

$\|\lambda Au\| = |\lambda| \|Au\|$

$\|\lambda Au\|^p = |\lambda|^p \|Au\|^p$

$\|\lambda Au\|^p \le g_{T(p)}(\lambda A)^p (\|u\|^p + \|Tu\|^p)$

$|\lambda|^p \| Au \|^p \le |\lambda|^p g_{T(p)}(A)^p (\|u\|^p + \|Tu\|^p)$.

Since $\|\lambda Au\|^p = |\lambda|^p \|Au\|^p$, the generalized$T(p)$-bounds are equal.

Hence, $g_{T(p)}(\lambda A)^p = |\lambda|^p g_{T(p)}(A)^p$.

Thus, $g_{T(p)}(\lambda A) = |\lambda| g_{T(p)}(A)$ for any scalarλ.

3. If $A$is a bounded operator, we have

$$\|Au\|^p \le \|A\|^p \|u\|^p \le \|A\|^p (\|u\|^p + \|Tu\|^p).$$

Hence $g_{T(p)}(A) \le \|A\|$.

Remark. Suppose $A$and $B$are $T-$bounded, where T is densely-defined in X. Then, since$D(T) \subset D(A)$ and $D(T) \subset D(B)$,$D(T) \subset D(A) \cap D(A)$. Hence, $A + B$is a densely-defined operator with $D(A + B) = D(A) \cap D(A)$.

Proposition 3.3. *Suppose Aand Bare T-bounded. Then*

$$g_{T(p)}(A + B) \le g_{T(p)}(A) + g_{T(p)}(B) \; . \qquad \dots (3.2)$$

Proof. For $u \in \mathrm{D}(T)$,$\|Au\|^p \le g_{T(p)}(A)^p (\|u\|^p + \|Tu\|^p)$

and$\|Bu\|^p \le g_{T(p)}(B)^p (\|u\|^p + \|Tu\|^p)$.

which implies,$\|Au\| \le g_{T(p)}(A) (\|u\|^p + \|Tu\|^p)^{1/p}$

and$\|Bu\| \le g_{T(p)}(B) (\|u\|^p + \|Tu\|^p)^{1/p}$.

Now,$\|(A + B)u\| \le \|Au\| + \|Bu\|$

$\le \left(g_{T(p)}(A) + g_{T(p)}(B)\right) (\|u\|^p + \|Tu\|^p)^{1/p}$.

Which implies, $\|(A + B)u\|^p \le \left(g_{T(p)}(A) + g_{T(p)}(B)\right)^p (\|u\|^p + \|Tu\|^p)$.

But, $\| (A + B)u \|^p \le g_{T_p}(A + B)^p (\|u\|^p + \| Tu \|^p)$.

Therefore $g_{T(p)}(A + B)^p \le \left(g_{T(p)}(A) + g_{T(p)}(B)\right)^p$.

Hence, $g_{T(p)}(A + B) \le g_{T(p)}(A) + g_{T(p)}(B)$.

Proposition 3.4. *Suppose Ais T-bounded and L is bounded , then* $g_{T(p)}(A + L) \le g_{T(p)}(A) + \| L \|$.

Proof. For $u \in \mathrm{D}(T)$,$\|(A + L)u\| \le \|Au\| + \|Lu\|$. ... (3.3)

Since A is$T$-bounded ,$\|Au\|^p \le g_{T(p)}(A)^p (\|u\|^p + \|Tu\|^p)$.

So, $\|Au\| \le g_{T(p)}(A) (\|u\|^p + \|Tu\|^p)^{1/p}$

and$\|Lu\|^p \le \|L\|^p \|u\|^p$

$\le \|L\|^p (\|u\|^p + \|Tu\|^p)$.

$\|Lu\| \le \|L\| (\|u\|^p + \|Tu\|^p)^{1/p}$.

Hence (3.3) implies, $\|(A + L)u\| \le g_{T_p}(A) (\|u\|^p + \|Tu\|^p)^{1/p} + \|L\| (\|u\|^p + \|Tu\|^p)^{1/p}$

$$= \left(g_{T(p)}(A) + \|L\|\right) (\|u\|^p + \|Tu\|^p)^{1/p}$$

$\|(A + L)u\|^p \le \left(g_{T(p)}(A) + \|L\|\right)^p (\|u\|^p + \|Tu\|^p)$.

Hence, $g_{T(p)}(A + L) \le g_{T(p)}(A) + \|L\|$.

4. Generalized p −approximation numbers

Definition 4.1 Let Tbe a closed operator in X and A be T-bounded. For $k = 1,2,3, \dots$, the k^{th}relative generalized$p-$approximation number $\theta_{k,T}^{(p)}(A)$ is defined as

$\theta_{k,T}^{(p)}(A) = inf\{g_{T(p)}(A - F)/F \in \mathcal{B}(X), rankF \le k - 1\}$. ...(4.1)

Also we define, $\theta_{\infty,T}^{(p)}(A) = inf\{g_{T(p)}(A - K)/K \in \mathcal{K}(X)\}$, ...(4.2)

where$\mathcal{K}(X)$ is the class of compact operators on X. It is clear from the definition that

$$g_{T(p)}(A) = \theta_{1,T}^{(p)}(A) \geq \theta_{2,T}^{(p)}(A) \geq \cdots \geq 0 \quad \ldots(4.3)$$

and$\theta_{k,T}^{(p)}(A) \geq \theta_{\infty,T}^{(p)}(A)$ for $k = 1, 2, 3, \ldots$ …(4.4)

Theorem 4.2 *Let A be T-bounded. Then*

$$\lim_{k\to\infty} \theta_{k,T}^{(p)}(A) = \inf_{k=1,2,3,\ldots} \theta_{k,T}^{(p)}(A) = \theta_{\infty,T}^{(p)}(A).$$

Proof. Let $m = \inf_{k=1,2,3,\ldots} \theta_{k,T}^{(p)}(A)$.

Then (4.4) implies that $m \geq \theta_{\infty,T}^{(p)}(A)$. …(4.5)

Now let $\varepsilon > 0$ and $K \in \mathcal{K}(X)$. Since the class of finite rank operators on X is dense in $\mathcal{K}(X)$, there is a finite rank operator F on X such that

$\|K - F\| < \varepsilon$. …(4.6)

For $u \in D(T)$, $\|(A-K)u\|^p \leq g_{T(p)}(A-K)^p[\|u\|^p + Tu^p]$.

Consider, $\|(A-F)u\|^p \leq [\|(A-K)u\| + \|(K-F)u\|]^p$

$\leq [\|(A-K)u\| + \varepsilon\|u\|]^p$ using(4.6)

$$\leq \left\{g_{T(p)}(A-K)[\|u\|^p + \|Tu\|^p]^{1/p} + \varepsilon[\|u\|^p + \|Tu\|^p]^{1/p}\right\}^p$$

$= \left[g_{T(p)}(A-K) + \varepsilon\right]^p [\|u\|^p + \|Tu\|^p]$.

Hence, $g_{T(p)}(A-F) \leq g_{T(p)}(A-K) + \varepsilon$.

This implies, $\theta_{k,T}^{(p)}(A) \leq g_{T(p)}(A-K) + \varepsilon$, if rank $F = k-1$.

So, $m \leq g_{T(p)}(A-K) + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, we get, $m \leq g_{T(p)}(A-K)$…(4.7)

(4.7) is the case for any $K \in \mathcal{K}(H)$.

Hence $m \leq g.l.b\left\{g_{T(p)}(A-K)/K \in \mathcal{K}(H)\right\} = \theta_{\infty,T}^{(p)}(A)$.

From (4.5) and (4.7), $m = \theta_{\infty,T}^{(p)}(A)$.

That is, $\inf_{k=1,2,3,\ldots} \theta_{k,T}^{(p)}(A) = \theta_{\infty,T}^{(p)}(A)$.

Again, since$\left\{\theta_{k,T}^{(p)}(A)\right\}_{k=1}^{\infty}$ is monotonic decreasing and bounded below,

$$\lim_{k\to\infty} \theta_{k,T}^{(p)}(A) = \inf_{k=1,2,3,\ldots} \theta_{k,T}^{(p)}(A) = \theta_{\infty,T}^{(p)}(A).$$

Definition 4.3. An operator$T$ is said to be semi-expansive if

$\|Tu\| \geq \|u\|$for every$u \in D(T)$ with $Tu \neq 0$. …(4.8)

Proposition 4.4. *Let A be T-bounded and L be bounded on X. Then,*

$\theta_{k,T}^{(p)}(A+L) \leq \theta_{k,T}^{(p)}(A) + \|L\|$ *for* $k = 1,2,3,\ldots$ *provided, T is semi-expansive.*

Proof. Let $k < \infty$and $F \in \mathcal{B}(X)$be with $rank F \leq k-1$. For any$u \in D(T)$, we have,

$\|(A-F)u\|^p \leq g_{T(p)}(A-F)^p[\|u\|^p + \|Tu\|^p]$. …(4.9)

Since $\|u\|^p + \|Tu\|^p \leq [\|u\| + \|Tu\|]^p$, (4.9) implies,

$\|(A-F)u\| \leq g_{T(p)}(A-F)[\|u\| + \|Tu\|]$.

Now consider, $\|(A+L-F)u\|^p \leq (\|(A-F)u\| + \|L\|\|u\|)^p$

$$\leq \|(A-F)u\|^p + pC_1\|(A-F)u\|^{p-1}\|L\|\|u\| + pC_2\|(A-F)u\|^{p-2}\|L\|^2\|u\|^2 + \ldots + \|L\|^p\|u\|^p$$

$$\leq g_{T(p)}(A-F)^p[\|u\|^p + \|Tu\|^p] + pC_1 g_{T(p)}(A-F)^{p-1}[\|u\|^{p-1} + \|Tu\|^{p-1}]\|L\|\|u\| + pC_2 g_{T(p)}(A-F)^{p-2}[\|u\|^{p-2} + \|Tu\|^{p-2}]\|L\|^2\|u\|^2 + \cdots + \|L\|^p\|u\|^p$$

$$\leq \left(g_{T(p)}(A-F) + \|L\|\right)^p\|u\|^p + g_{T(p)}(A-F)^p\|Tu\|^p + pC_1 g_{T(p)}(A-F)^p\|Tu\|^{p-1}\|L\|\|u\| + pC_2 g_{T(p)}(A-F)^{p-2}\|Tu\|^{p-2}\|L\|^2\|u\|^2 + \cdots +$$

$pC_{p-1} g_{T(p)}(A-F)\|Tu\|\|L\|^{p-1}\|u\|^{p-1}, u \in D(T)$…(4.10)

As $T$is semi-expansive, using (4.8), for$u \in D(T)$ with $Tu \neq 0$, (4.10) becomes,

$$\|(A+L-F)u\|^p \leq \left(g_{T(p)}(A-F) + \|L\|\right)^p + g_{T_p}(A-F)^p\|Tu\|^p + pC_1 g_{T(p)}(A-F)^p\|Tu\|^p\|L\| + pC_2 g_{T(p)}(A-F)^{p-2}\|Tu\|^p\|L\|^2 + \cdots + pC_{p-1} g_{T(p)}(A-F)\|Tu\|^p$$

$$\leq \left(g_{T(p)}(A-F) + \|L\|\right)^p\|u\|^p + \left(g_{T(p)}(A-F)^p + \|L\|\right)^p\|Tu\|^p$$

$= \left(g_{T(p)}(A-F) + \|L\|\right)^p(\|u\|^p + \|Tu\|^p)$. …(4.11)

(4.11) is valid even if $Tu = 0$. Compare with (4.10). Thus (4.11) is the case for every $u \in D(T)$.

Hence$g_{T(p)}(A+L-F) \leq g_{T(p)}(A-F) + \|L\|$,

which implies that, $\theta_{k,T}^{(p)}(A+L) \le g_{T(p)}(A-F) + \|L\|$. ...(4.12)
Since (4.12) is true for any $F \in \mathcal{B}(X)$ with rank $F \le k-1$, we have,
$\theta_{k,T}^{(p)}(A+L) \le \theta_{k,T}^{(p)}(A) + \|L\|$.
The proof is similar for the case $k = \infty$.

Proposition 4.5. *If Ais a bounded operator,* $\theta_{k,T}^{(p)}(A) \le a_k(A)$.

Proof. $\theta_{k,T}^{(p)}(A) = g.l.b\{g_{T(p)}(A-F)/F \in \mathcal{B}(X), rankF \le k-1\}$
and since A is bounded operator, $g_{T(p)}(A-F) = \|A-F\|$.

Theorem 4.6. *Let Aand Bbe T-bounded. Then*
$\theta_{k_1+k_2-1,T}^{(p)}(A+B) \le \theta_{k_1,T}^{(p)}(A) + \theta_{k_2,T}^{(p)}(B)$.

Proof. Let $F_1, F_2 \in \mathcal{B}(X)$ with $rankF_1 \le k_1 - 1$ and $rankF_2 \le k_2 - 1$.
Then $F_1 + F_2 \in \mathcal{B}(X)$ with $rankF_1 + F_2 \le k_1 + k_2 - 2$.
Consider $g_{T(p)}(A+B-F_1-F_2) \le g_{T(p)}(A-F_1) + g_{T(p)}(A-F_2)$.
Hence, $\theta_{k_1+k_2-1,T}^{(p)}(A+B) \le g_{T(p)}(A-F_1) + g_{T(p)}(A-F_2)$. ...(4.13)
Since (4.13) is valid for any $F_1, F_2 \in \mathcal{B}(X)$ with $rankF_1 \le k_1 - 1$ and
$rankF_2 \le k_2 - 1$, we have, $\theta_{k_1+k_2-1,T}^{(p)}(A+B) \le \theta_{k_1,T}^{(p)}(A) + \theta_{k_2,T}^{(p)}(B)$.

5. Generalized p −approximation numbers using a new norm.

Let Xbe a Banach space and$T$be a densely defined closed linear operator in X. Define $\| \ \|_{T(p)}$ on $D(T)$ by $\|u\|_{T(p)} = (\|u\|^p + \|Tu\|^p)^{\frac{1}{p}}$.

Remark. $\| \ \|_{T(p)}$*is a norm on* $D(T)$.We denote the space $D(T)$ with the norm $\| \ \|_{T(p)}$ by $X_{T(p)}$.

Proposition 5.1 $X_{T(p)}$*is abanach space.*

Proof. Let (x_n)be a Cauchy sequence in $X_{T(p)}$.

For $n, m \in \mathbb{N}$, $\|u_n - u_m\|_{T(p)} = (\|u_n - u_m\|^p + \|Tu_n - Tu_m\|^p)^{\frac{1}{p}}$.
Hence, both (u_n) and (Tu_n) are Cauchy sequences in X.
Let $(u_n) \to u$ and $(Tu_n) \to v$. Since $T$is closed, $u \in D(T) = X_{T(p)}$ and $Tu = v$.

$$\text{Hence } \|u_n - u\|_{T(p)} = (\|u_n - u\|^p + \|Tu_n - Tu\|^p)^{\frac{1}{p}}$$
$$= (\|u_n - u\|^p + \|Tu_n - v\|^p)^{\frac{1}{p}}$$
$\to 0$ as $n \to \infty$.
Thus $(u_n) \to$ u in $X_{T(p)}$ and $X_{T(p)}$ is complete.

Remark. A is $T(p)$-bounded if and only if there are non-negative reals m, $n$such that $\|Au\|^p \le m^p\|u\|^p + n^p\|Tu\|^p$ for every $u \in D(T)$if and only if
$\|Au\|^p \le \alpha^p(\|u\|^p + \|Tu\|^p)$ for every $u \in D(T)$, for some non-negative real α.
Thus, $A$is $T(p)$-bounded if and only if $\|Au\|^p \le \alpha^p\|u\|_{T(p)}^p$ for every $u \in D(T)$
If and only if $\|Au\| \le \alpha\|u\|_{T(p)}$for every $u \in D(T)$.
We observethat the $T(p)$-boundednessis equivalent, to the generalized relative $T(p)$-boundedness namely, A is generalized $T(p)$-bounded if and only if there exists non-negative real number $\alpha < \infty$ such that

$$\|Au\| \le \alpha(\|u\|^p + \|Tu\|^p)^{\frac{1}{p}}, \qquad u \in D(T).$$

Suppose A is a $T(p)$-bounded operator. Let $\hat{A}$ be the restriction of A to $D(T)$. Then $\hat{A}$ can be considered as an operator on $X_{T(p)}$:

$$\hat{A}u = Au, \quad u \in X_{T(p)} = D(T).$$

Proposition 5.2 $\hat{A}$ *is a bounded operator. In this case* $\|\hat{A}\| = g_{T(p)}(A)$.

Proof. Since$A$is $T(p)$-bounded, there is $\alpha < \infty$ such that

$$\|Au\| \le \alpha(\|u\|^p + \|Tu\|^p)^{\frac{1}{p}}, \qquad u \in D(T)$$
$$= \alpha\|u\|_{T(p)}, \quad u \in X_{T(p)}.$$

i.e. $\|\hat{A}u\| \le \alpha\|u\|_{T(p)}, \qquad u \in X_{T(p)}$.
Hence $\hat{A}$ is a bounded operator.

$$\text{Now, } \|\hat{A}\| = \inf\{\alpha : \|\hat{A}u\| \le \alpha \text{ for all } u \in X_{T(p)}\}$$
$$= \inf\left\{\alpha : \|\hat{A}u\|^p \le \alpha^p\|u\|_{T(p)}^p \text{ for all } u \in X_{T(p)}\right\}$$
$$= \inf\{\alpha : \|Au\|^p \le \alpha^p(\|u\|^p + \|Tu\|^p) \text{ for all } u \in D(T)\}$$

$= g_{T(p)}(A)$.

Definition 5.3

Let A be a $T(p)$-bounded operator and $k \in \mathbb{N}$. Then the k^{th}relative p-approximation number of $A$with respect to$T$, or the $T(p)$-approximation number of A is defined by $\hat{s}_{k,p}(A) = s_k(\hat{A})$, where $s_k(\hat{A})$ denotes the k^{th} approximation number of the bounded operator $\hat{A}$: $s_k(\hat{A}) = \inf\{\|\hat{A} - F\| : F \in \mathcal{B}(X),\ rank\ F \leq k-1\}$

Proposition 5.4 *Suppose A_1, $A_2$are $T(p)-$ bounded. Then $A_1 + A_2$ is $T(p)$-bounded and*
$\widehat{A_1 + A_2} = \widehat{A_1} + \widehat{A_2}$ *on* $X_{T(p)}$,
$\|\widehat{A_1 + A_2}\| \leq \|\widehat{A_1}\| + \|\widehat{A_2}\|$.

Proof. For $u \in X_{T(p)}$, $\|A_1 u\| \leq \alpha \|u\|_{T(p)}$ and $\|A_2 u\| \leq \beta \|u\|_{T(p)}$.

Now $\|(A_1 + A_2)u\| \leq \|A_1 u\| + \|A_2 u\|$
$\leq \alpha \|u\|_{T(p)} + \beta \|u\|_{T(p)}$
$= (\alpha + \beta)\|u\|_{T(p)}$
$= \gamma \|u\|_{T(p)}$, where $\gamma = \alpha + \beta$.

Hence $A_1 + A_2$ is a $T(p)-$bounded.

Also for $u \in X_{T(p)}$, $(\widehat{A_1 + A_2})u = (A_1 + A_2)u$
$= A_1 u + A_2 u$
$= \widehat{A_1} u + \widehat{A_2} u$
$= (\widehat{A_1} + \widehat{A_2})u$.

Hence$\widehat{A_1 + A_2} = \widehat{A_1} + \widehat{A_2}$on $X_{T(p)}$.

$$\|(\widehat{A_1 + A_2})u\| = \|(\widehat{A_1} + \widehat{A_2})u\|$$
$$\leq \|\widehat{A_1} u\| + \|\widehat{A_2} u\|$$
$\leq (\|\widehat{A_1}\| + \|\widehat{A_2}\|)\|u\|$.
$\|\widehat{A_1 + A_2}\| \leq \|\widehat{A_1}\| + \|\widehat{A_2}\|$.

Proposition 5.5*IfA is bounded, then A is $T(p)$-bounded , and $\|\hat{A}\| \leq \|A\|$.*

Proof. If A is bounded, then $D(A) = X$.
Also for $u \in D(T)$,$\hat{A}u = Au$.
$\|\hat{A}u\| = \|Au\| \leq \|A\|\|u\| \leq \|A\|\|u\|_{T(p)}$.
Hence A is $T(p)-$bounded and $\|\hat{A}\| \leq \|A\|$.

Proposition 5.6 *IfA is $T(p)$-bounded and B is a bounded operator, then BA is T-bounded ,$\widehat{BA} = B\hat{A}$ and $\|\widehat{BA}\| = \|B\|\|\hat{A}\|$.*

Proof. If A is $T(p)$-bounded and B is bounded, then $D(BA) = D(A)$. Also for $u \in X_{T(p)}$, we have$\widehat{BA}(u) = B(\hat{A}u)$.
Hence $\|\widehat{BA}\| = \|B\|\|\hat{A}\|$.

Proposition 5.7 *Let A be $T(p)$-bounded , invertible and A^{-1}be a bounded operator, then $\hat{A}$is invertible, $(\hat{A})^{-1} = \widehat{A^{-1}}$, and $\|(\hat{A})^{-1}\| \leq (\|A^{-1}\|^p + \|TA^{-1}\|^p)^{\frac{1}{p}}$*
In particular, if T itself satisfies the above hypothesis, then
$\|(\hat{T})^{-1}\| \leq (1 + \|T^{-1}\|^p)^{\frac{1}{p}}$.

Proof. Since A^{-1} is a bounded operator, TA^{-1} is a bounded operator.
Also,$(\hat{A})^{-1} = \widehat{A^{-1}}$, and for $z \in Z$,

$$\|(\hat{A})^{-1} z\|_{T(p)} = \|\widehat{A^{-1}} z\|_{T(p)} = \|A^{-1}z\|_{T(p)} = (\|A^{-1}z\|^p + \|TA^{-1}z\|^p)^{\frac{1}{p}}$$
$$\leq (\|A^{-1}\|^p\|z\|^p + \|TA^{-1}\|^p\|z\|^p)^{\frac{1}{p}}$$
$$= \{(\|A^{-1}\|^p + \|TA^{-1}\|^p)\|z\|^p\}^{1/p}$$
$= (\|A^{-1}\|^p + \|TA^{-1}\|^p)^{\frac{1}{p}}\|z\|$.

Hence $\|(\hat{A})^{-1}\| \leq (\|A^{-1}\|^p + \|TA^{-1}\|^p)^{\frac{1}{p}}$.

In particular, taking T in place of$A$, we get,

$$\|(\hat{T})^{-1}\| \leq (\|T^{-1}\|^p + \|TT^{-1}\|^p)^{\frac{1}{p}}$$
$= (\|T^{-1}\|^p + 1)^{\frac{1}{p}}$.

Proposition 5.8 *For $k \in \mathbb{N}$, the following properties of $T(p)$-approximation number hold.*

(a) $\hat{s}_{k,p}(T) \leq 1$.
(b) $\hat{s}_{k,p}(B) \leq s_k(B)$ *if B is a bounded operator.*
(c) $\hat{s}_{k_1+k_2-1,p}(A_1 + A_2) \leq \hat{s}_{k_1,p}(A_1) + \hat{s}_{k_2,p}(A_2)$ *for $T(p)$-bounded operators* A_1, A_2
(d) $\hat{s}_{k,p}(BA) \leq \|B\|\hat{s}_{k,p}(A)$ *if A is $T(p)$-bounded and B is a bounded operator.*

Proof:

(a) $\hat{s}_{k,p}(T) = s_k(\hat{T}) = \inf\{\|\hat{T} - F\|: F \in \mathcal{B}(X),\ rank\ F \leq k-1\} \leq \|\hat{T}\| \leq 1$.

(b) $\hat{s}_{k,p}(B) = s_k(\hat{B}) \leq \|\hat{B}\|$
and $s_k(B) \leq \|B\|$
Since B is bounded, $\|\hat{B}\| \leq \|B\|$

$$\hat{s}_{k,p}(B) \leq s_k(B)$$

(c) Let A_1, A_2 be $T(p)$-bounded operators. Let F_1, F_2 be a bounded linear operator on $X_{T(p)}$ such that rank $(F_1) \leq k_1 - 1$ and rank $(F_2) \leq k_2 - 1$. Then $F_1 + F_2$ be a bounded linear operator on $X_{T(p)}$ with $k = k_1 + k_2 - 1$ and

$$\|(\widehat{A_1 + A_2}) - (F_1 + F_2)\| = \|\widehat{A_1} + \widehat{A_2} - F_1 - F_2\|$$
$$= \|(\widehat{A_1} - F_1) + (\widehat{A_2} - F_2)\|$$
$$\leq \|\widehat{A_1} - F_1\| + \|\widehat{A_2} - F_2\|$$
$$\hat{s}_{k_1+k_2-1,p}(A_1 + A_2) = \hat{s}_{k,p}(A_1 + A_2) = s_k(\widehat{A_1 + A_2})$$
$$= \inf\{\|(\widehat{A_1 + A_2}) - (F_1 + F_2)\|:\ F_1 + F_2 \in \mathcal{B}(X_{T(p)}, X), rank\ (F_1 + F_2) \leq k-1\}$$
$$\leq \|(\widehat{A_1 + A_2}) - (F_1 + F_2)\|$$
$$\leq \|\widehat{A_1} - F_1\| + \|\widehat{A_2} - F_2\|$$
$$\hat{s}_{k_1+k_2-1,p}(A_1 + A_2) \leq \inf\{\|\widehat{A_1} - F_1\|: F_1 \in \mathcal{B}(X_{T(p)}, X), rank\ (F_1) \leq k_1 - 1\}$$
$$+\inf\{\|\widehat{A_2} - F_2\|: F_2 \in \mathcal{B}(X_{T(p)}, X), rank\ (F_2) \leq k_2 - 1\}$$
$$\leq \hat{s}_{k_1,p}(A_1) + \hat{s}_{k_2,p}(A_2)$$

(d) Let A be a $T(p)$-bounded operator, B be a bounded linear operator , and let F be a bounded linear operator on $X_{T(p)}$ with $rank(F) \leq k-1$. Then by proposition, BA is a $T(p)$-bounded operator with $D(\mathrm{BA}) = D(\mathrm{A})$ and BF is a bounded linear operator on $X_{T(p)}$ with $rank(F) \leq k-1$.
Since $\|\widehat{BA}\| = \|B\|\|\hat{A}\|$.
$\|(\widehat{BA} - BF)u\| \leq \|B\|\|(\hat{A} - F)u\|, \quad u \in X_{T(p)}$.
We have $\|\widehat{BA} - BF\| \leq \|B\|\|\hat{A} - F\|$.
Hence we obtain $\hat{s}_{k,p}(BA) \leq \|B\|\hat{s}_{k,p}(A)$.

Proposition 5.9 *Let A be $T(p)$-bounded, invertible and A^{-1} be a bounded linear operator. Then*

$$\hat{s}_{k,p}(A) \geq \frac{1}{(\|A^{-1}\|^p + \|TA^{-1}\|^p)^{\frac{1}{p}}} \quad \forall\, k \in \mathbb{N}$$

In particular, if T is invertible and T^{-1} is a bounded linear operator, then

$$\hat{s}_{k,p}(T) \geq \frac{1}{1 + \|T^{-1}\|} \quad \forall\, k \in \mathbb{N}$$

Proof. We have, for all $k \in \mathbb{N}$, from proposition 5.7,

$$1 = s_k\left(I_{X_{T(p)}}\right) = s_k\left(\left(\hat{A}\right)^{-1}\hat{A}\right) \leq s_k(\hat{A})\left\|\left(\hat{A}\right)^{-1}\right\| \leq \hat{s}_{k,p}(A)(\|A^{-1}\|^p + \|TA^{-1}\|^p)^{\frac{1}{p}}$$

$$\hat{s}_{k,p}(A) \geq \frac{1}{(\|A^{-1}\|^p + \|TA^{-1}\|^p)^{\frac{1}{p}}} .$$

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