ON SOFT γ-CONNECTED SPACES IN SOFT TOPOLOGICAL SPACES

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Abstract: In this paper, based on the concept of soft γ-operations on soft topological spaces , the notions of soft γ-connected spaces in soft topological spaces are introduced and the related properties are studied.

Keywords: soft set, soft topology, soft γ-operation, soft γ-open set, soft connected space soft γ-connected space.

1. Introduction: The soft set theory is introduced by Molodtsov [1] in 1999, as a new mathematical tool and shown how soft set theory is better than fuzzy set theory, rough set theory and game theory. Further, Maji et al [2] introduced and studied several basic definitions and basic operations of soft sets. Soft topological spaces are introduced by Shabir and Naz [3] , which are defined over an initial universe with a fixed set of parameters and studied the basic notions such as soft open sets, soft closed sets, soft closure, soft separation axioms. Sabir and Bashir [4] and Naim Cagman et al [5] continued the study of properties of soft topological spaces. The study of soft sets and related aspects was also undertaken in [6],[7],[8],[9]. Recently, many researchers have introduced various weaker forms of soft open sets and soft closed sets in soft topological spaces and studied their properties in [10], [11], [12], [13], [14], [15], [16], [17]. The concept of soft γ-operations, soft γ-compact and soft γ-normal spaces in soft topological space are introduced and studied by S.S.Benchalli et al [18], [19]. In this paper, based on the concept of soft γ-operations on soft topological spaces , the notions of soft γ-connected spaces in soft topological spaces are introduced and the related properties are studied.

2.Preliminaries: The following preliminaries are required for subsequent work.

Definition 2.1.[1]: Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F,A) is called a soft set over U ,where F is a mapping given by $F: A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A, F(e)$ may be considered as the set of e-approximate elements of the soft set (F,A) . Clearly, every set is a soft set but not conversely.

Definition 2.2. [2]: For two soft sets (F,A) and (G,B) over a common universe , we say that (F,A) is a soft subset of (G,B) if (i)$A \subset B$ and (ii) for all $e \in A, F(e)$ and $G(e)$ are identical approximations.

Definition 2.3.[3]: Let τ be the collection of soft sets over X. Then τ is said to be a soft topology on X if

(1) $\emptyset, \tilde{X}$ belongs to τ.

(2) The union of any number of soft sets in τ belongs to τ.

(3) The intersection of any two soft sets in τ belongs to τ.

The triplet (X,τ,E) is called a soft topological space. Here the members of τ are called soft open sets in X and the relative complements of soft open sets are called soft closed sets.

Theorem 2.4. [3]: Arbitrary union of soft open sets is a soft open set and finite intersection of soft closed sets is a soft closed set.

Definition 2.5.[3]: Let (X,τ,E) be a soft space over X and (F,E) be a soft set over X. Then,the soft closure of (F,E) denoted by $\overline{(F,E)}$ is the intersection of all soft closed super sets of (F,E).Clearly, $\overline{(F,E)}$ is the smallest soft closed set over X containing (F,E).

The soft neighborhood, soft relative topology, soft T_0- spaces, soft T_1- spaces and soft T_2- spaces are defined and studied by Shabir and Naz in [6].

Definition 2.6.[9] : The soft interior of (G,E) is the soft set defined as

$(G,E)^o$ = $\mathrm{Int}(G,E) = \cup\{(S,E):(S,E)$ is soft open and $(S,E) \sqsubseteq (G,E)\}$. Here $(G,E)^o$ is largest soft open set contained in (G,E).

Definition 2.7.[18]: Let (X,τ,E) be a soft topological space. An operation γ on the soft topology τ is a mapping from τ into the power set $P(X)$ of X such that $(V,E) \subset (V,E)^\gamma$, for each $(V,E) \in \tau$, where $(V,E)^\gamma = \gamma(V,E)$. It is denoted by $\gamma: \tau \to P(X)$.

Definition 2.8.[18]: A subset (A,E) of a soft topological space (X,τ,E) is called a soft γ-open set of (X,τ,E), if for each $x \in (A,E)$ there exists a soft open set (U,E) such that $x \in (U,E) \subset (U,E)^\gamma \subset (A,E)$. τ_γ will denote the set of all soft γ-open sets. Clearly, we have $\tau_\gamma \subset \tau$.

A subset (B,E) of (X,τ,E) is called soft γ-closed if $(B,E)'$ is soft γ-open in (X,τ,E).

Definition 2.9.[18]: A point $x \in X$ is called a soft γ-closure point of (A,E), if $(U,E)^\gamma \cap (A,E) \neq \phi$ for each soft open neighbourhood(nbd) (U,E) of x. The set of soft γ-closure points is called the soft γ-closure of (A,E) and is denoted by $Cl_\gamma(A,E)$.

For the family τ_γ, we define a soft set $\tau_\gamma - Cl(A,E)$ as

$$\tau_\gamma - Cl(A,E) = \cap\{(F,E)/(F,E) \supset (A,E) and (F,E)' \in \tau\}$$

Definition 2.10. [18]: An operation γ on τ is said to be soft open if for every soft nbd (U,E) of each of

$x \in X$, there exists a soft γ-open set (B,E) such that $x \in (B,E) \subseteq (U,E)^{\gamma}$.

Definition 2.11. [18]: An operation γ on τ is said to be soft regular if for any soft nbds (U,E) and (V,E) of $x \in X$, there exists soft open nbd (W,E) of x such that $(W,E)^{\gamma} \subseteq (U,E)^{\gamma} \cap (V,E)^{\gamma}$.

Definition 2.12. [3]: A soft topological space (X,τ,E) is called soft γ-regular if for each soft open nbd (U,E) of x in X, there exists a soft open nbd (V,E) of x such that $(V,E)^{\gamma} \subseteq (U,E)$.

Theorem 2.13.[3]: For any soft sets $(A,E),(B,E)$ in soft topological space (X,τ,E) the following hold:

(a)$Cl_{\gamma}[(A,E)-(B,E)] \supseteq Cl_{\gamma}(A,E) - Cl_{\gamma}(B,E)$

(b)$Int_{\gamma}[(A,E)-(B,E)] \subseteq Int_{\gamma}(A,E) - Int_{\gamma}(B,E)$

(c)If (A,E) is soft γ-open, then $(A,E) \cap Cl_{\gamma}(B,E) \subseteq Cl_{\gamma}(B,E) \subseteq Cl_{\gamma}((A,E) \cap (B,E))$

Proposition 2.14. [3]: If γ is soft regular then $Cl_{\gamma}((A,E) \cup (B,E)) = Cl_{\gamma}(A,E) \cup Cl_{\gamma}(B,E)$.

Definition 2.15. [14]: A soft topological spaces (X,τ,E) is called soft connected space if there do not exists a pair $(A,E),(B,E)$ of non empty disjoint soft open subsets of (X,τ,E) such that $(X,E)=(A,E)\cup(B,E)$, otherwise (X,τ,E) is said to be soft disconnected space.

Definition 2.16. [3]: The soft γ-exterior of $(A,E)$is defined as the soft γ-interior of $(A,E)'$.

That is, $ext_{\gamma}(A,E) = \mathrm{Int}_{\gamma}(A,E)'$.

Definition 2.17. [3]: The soft γ-boundary of (A,E) is denoted by $bd_{\gamma}(A,E)$, is defined as the set of all points which do not belongs to soft γ-interior or soft γ-exterior of (A,E).

Theorem 2.18. [18]: In any soft topological space (X,τ,E) the following is true,

$bd_{\gamma}(A,E) = Cl_{\gamma}(A,E) \cap Cl_{\gamma}(A,E)' = Cl_{\gamma}(A,E) - Int_{\gamma}(A,E)$.

3. Soft γ-connected spaces: Now we define and discuss a new space called soft γ-connected space in a soft topological space (X,τ,E). It is remarkable that the class of soft connected spaces is the subclass of soft γ-connected spaces.

Definition 3.1. A soft topological space (X,τ,E) is said to be soft γ-connected if there does not exist non empty soft γ-open sets $(A,E),(B,E)$ of (X,τ,E) such that $(X,E) = (A,E) \cup (B,E)$, $(A,E) \cap Cl_{\gamma}(B,E) = \emptyset, Cl_{\gamma}(A,E) \cap (B,E) = \emptyset$.

Here, $((A,E),(B,E))$ is called soft γ-disconnection of (X,τ,E).

Example 3.2. Let $X = \{a,b,c\}$, $E = \{e_1,e_2\}$, $\tau = \{\emptyset, X, (A,E),(B,E),(C,E),(D,E)\}$ where $(A,E) = \{(e_1,\{a\}),(e_2,\{a\})\}$,

$(B,E) = \{(e_1,\{b\}),(e_2,\{b\})\}$

$(C,E) = \{(e_1,\{a,b\}),(e_2,\{a,b\})\}$,

$(D,E) = \{(e_1,\{a,c\}),(e_2,\{a,c\})\}$

For $b \in X$, define an operation $\gamma: \tau \to P(X)$ as

$$\gamma(A,E) = \begin{cases} Cl(A,E), & if b \in (A,E) \\ Cl(Int(A,E)), & if b \notin (A,E) \end{cases}$$

We can verify that, $\emptyset,(X,E),(D,E)$ are the only soft γ-open sets. Clearly , (X,τ,E) is the soft γ-connected space but not soft connected space.

Theorem 3.3. A soft topological space (X,τ,E) is soft γ-disconnected if and only if there exists non empty proper subset (A,E) of (X,τ,E) which is both soft γ-open and soft γ-closed in (X,τ,E), where γ is regular operation.

Proof. Given (A,E) is both soft γ-open and soft γ-closed. Now, (A,E) is soft γ-open, which implies $(A,E)' = (B,E)$ (say) is soft γ-closed. Since (B,E) is soft complement of (A,E) in (X,τ,E), therefore $(A,E) \cup (B,E) = (X,E)$ and also $(A,E) \cap (B,E) = \emptyset$. Also, $(A,E) \cap Cl_{\gamma}(B,E) = \emptyset, Cl_{\gamma}(A,E) \cap (B,E) = \emptyset$. Thus, by definition, (X,τ,E) is soft γ-disconnected space.

Conversely, let (X,τ,E) be a soft γ-disconnected space, so that there exists two non empty proper subsets (A,E),(B,E) such that $(A,E) \cup (B,E) = (X,E)$, $(A,E) \cap Cl_{\gamma}(B,E) = \emptyset$, $Cl_{\gamma}(A,E) \cap (B,E) = \emptyset$. We know that $(A,E) \subset Cl_{\gamma}(A,E)$. Thus, $Cl_{\gamma}(A,E) \cap (B,E) = \emptyset$ implies $(A,E) \cap (B,E) = \emptyset$. As $(A,E) \cup (B,E) = (X,E)$ and $(A,E) \cap (B,E) = \emptyset$, we get $(A,E) = (B,E)'$. Thus, (A,E) is a proper subset of (X,τ,E) which is non empty. Again, $(B,E) \subset Cl_{\gamma}(B,E)$, therefore $(A,E) \cup (B,E) = (X,E)$ implies $(A,E) \cap Cl_{\gamma}(B,E) = X$ and $(A,E) \cap (B,E) = \emptyset$ implies $(A,E) \cap Cl_{\gamma}(B,E) = \emptyset$. Thus we get $(A,E) = \left(Cl_{\gamma}(B,E)\right)'$. But as $Cl_{\gamma}(B,E)$ is a soft γ-closed set, we have (A,E) is a soft γ-open set. In a similar manner, we can show that $(B,E) = \left(Cl_{\gamma}(A,E)\right)'$ is soft γ-open set. Now, $(A,E) = (B,E)'$, where (B,E) is soft γ-open set and hence (A,E) is soft γ-closed. Thus, (A,E) is both soft γ-open and soft γ-closed set in (X,τ,E). This completes the proof.

Theorem 3.4. A soft topological space (X,τ,E) is soft γ-connected if and only if there exists a non empty subset of (X,E) which is both soft γ-open and soft γ-closed in (X,E) itself.

Proof: Given (X,τ,E) is soft γ-connected and (A,E) is non empty subset which is both soft γ-open and soft γ-closed. Now, it is enough to prove that (A,E)=(X,E). Since (A,E) is both soft γ-open and soft γ-closed , then $(A,E)'$ is both soft γ-open and soft γ-closed. This implies $(A,E) = Cl_{\gamma}(A,E)$, $(A,E)' = Cl_{\gamma}(A,E)'$. Also, $(A,E) \cap (A,E)' = \emptyset$ and $(A,E) \cup (A,E)' = (X,E)$. Hence, we have (X,τ,E) is soft γ-disconnected space. This contradicts the fact that (X,τ,E) is soft γ-connected space. Therefore there must have either of (A,E) or $(A,E)'$ as empty. But since (A,E) is chosen to be non empty, therefore $(A,E)' = \emptyset$. Thus

$(X,E)=(A,E)\cup(A,E)'=(A,E)\cup\emptyset=(A,E)$. Hence, the non empty subset which is both soft γ-open and soft γ-closed is (X,E) itself.

Conversely, given that only non empty subset which is both soft γ-open and soft γ-closed is (X,E) itself. That is, there do not exists any proper subset of (X,E) which is both soft γ-open and soft γ-closed and hence by theorem 3.3, (X,τ,E) is soft γ-connected space. This completes the proof.

Example 3.5. Every soft indiscrete topological space (X,I,E), where (X,E) is non empty, is soft γ-connected.

Theorem 3.6. A soft topological space (X,τ,E) is soft γ-connected if and only if every non empty proper subspace has a non empty soft γ-boundary.

Proof: Suppose that a non empty proper subspace (A,E) of a soft γ-connected space (X,τ,E) has empty soft γ-boundary. Then, by theorem 2.22, we have $bd_\gamma(A,E)=Cl_\gamma(A,E)-Int_\gamma(A,E)=\emptyset$. This implies $Cl_\gamma(A,E)=Int_\gamma(A,E)$ and we know that $Int_\gamma(A,E)\sqsubset$ (A, E).Thus, $Cl_\gamma(A,E)\sqsubset$ (A, E). But $(A,E)\sqsubset Cl_\gamma(A,E)$ is always true. Then, $(A,E)=Cl_\gamma(A,E)$ and (A,E) is soft γ-closed. Further we have $(A,E)\sqsubset Cl_\gamma(A,E)=Int_\gamma(A,E)$, which implies $(A,E)\sqsubset Int_\gamma(A,E)$. But $Int_\gamma(A,E)\sqsubset$ (A, E) is always true. Thus, we have $Int_\gamma(A,E)=$ (A, E) and (A,E) is soft γ-open. Hence, (A,E) is both soft γ-open and soft γ-closed. By theorem 3.3, (X,τ,E) is soft γ-disconnected space. Which is a contradiction. Thus, (A,E) has a non empty soft γ-boundary.

Conversely, suppose (X,τ,E) is soft γ-disconnected space. Then, by theorem 3.3, (X,τ,E) has a proper subspace (A,E) which is both soft γ-open and soft γ-closed. Then, $(A,E)=Cl_\gamma(A,E)$ and $(A,E)'=Cl_\gamma(A,E)'$ and $Cl_\gamma(A,E)\cap Cl_\gamma(A,E)'=\emptyset$. So, (A,E) has a empty soft γ-boundary, which is a contradiction. Thus, (□,□,□) is soft □-connected space. This completes the proof.

Theorem 3.7. Let $((A,E),(B,E))$ be a soft γ-disconnection of a soft topological space (X,τ,E) and (C,E) be a soft γ-connected subspace of (X,τ,E). Then, (C,E) is contained in $(A,E) or (B,E)$.

Proof: Suppose that (C,E) is neither contained in $(A,E) nor in (B,E)$. Then, $(C,E)\cap(A,E)$ and $(C,E)\cap(B,E)$ are both non empty soft γ-open subsets of (C,E) such that $[(C,E)\cap(A,E)]\cap[(C,E)\cap(B,E)]=\emptyset$ and $[(C,E)\cap(A,E)]\cup[(C,E)\cap(B,E)]=(C,E)$. Thus, $((C,E)\cap(A,E),(C,E)\cap(B,E))$ is soft γ-disconnection of (C,E). This is a contradiction to the hypothesis that (C,E) is soft γ-connected. Hence, Then, (C,E) is contained in $(A,E) or (B,E)$.

Theorem 3.8. Let $(X,E)=\cup_{\alpha\in I}(X,E)_\alpha$,where each $(X,E)_\alpha$ is soft γ-connected and $\cap_{\alpha\in I}(X,E)_\alpha\neq\emptyset$. Then, (X,E) is soft γ-connected.

Proof: Consider the contradiction that $((A,E),(B,E))$ is a soft γ-disconnection of (X,E). Since, each $(X,E)_\alpha$ is soft γ-connected, therefore from theorem 3.7, $(X,E)_\alpha\sqsubset$ (A, E) or $(X,E)_\alpha\sqsubset$ (B, E). Since, $\cap_{\alpha\in I}(X,E)_\alpha\neq\emptyset$, therefore all $(X,E)_\alpha$are contained in $(A,E) or (B,E)$. This implies if $(X,E)\sqsubset$ (A, E) then (B, E) $=\emptyset$ or if $(X,E)\sqsubset$ (B, E) then (A, E) $=\emptyset$. But, it is a contradiction to the hypothesis that (X,E) is soft γ-disconnected. Thus, (X,E) is soft γ-connected. This completes the proof.

Theorem 3.9. A soft topological space (X,τ,E) is soft γ-connected if and only if for every pair of points x,y in (X,τ,E), there is a soft γ-connected subspace of (X,τ,E) which contains both x and y.

Proof : The necessary condition is straight forward since the soft γ-connected space itself contains these two points.

For sufficient condition, let us consider for two points x and y, there is a soft γ-connected subspace $(C,E)_{xy}$ of (X,τ,E) such that $x,y\in(C,E)_{xy}$. Let $a\in(X,E)$ be a fixed point and $\{(C,E)_{ax}: x\in X\}$ be a set of all soft γ-connected subsets of (X,τ,E) which contains $a,x\in(X,E)$. Then, $(X,E)=\cup(C,E)_{ax}$ and $\cap_{x\in X}(C,E)_{ax}\neq\emptyset$. Therefore, by theorem 3.8, (X,τ,E) soft γ-connected space. This completes the proof.

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