FUZZY ALGEBRAIC STRUCTURE IN BP – ALGEBRAS

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Abstract: In this paper, we define the notion of Fuzzy BP-Algebras. We discuss the properties of Fuzzy BP-subalgebras and prove some results. The notion of intersection of fuzzy BP-subalgebras, Cartesian product of fuzzy BP-subalgebras.

Keywords: BP-algebra, Fuzzy BP-algebra

Introduction: In 1966 Y.Imai and K.Iseki introduced two classes of abstract algebra, BCK algebras and BCI algebras [1,2]. In 2012 Sun Shin Ahn and Jeong Soon Han introduced the notion of BP-Algebras[3]. In 1965 L.A.Zadeh[5] introduced the notion of fuzzy sets, in which the boundaries are not crisp or sharp. The study of fuzzy algebraic structures was initiated by A.Rosenfeld [4]. In this paper, we introduce the notion of Fuzzy BP-subalgebras.

Preliminaries

Definition 2.1 A BCK-algebra (X, ∗,0) is a non empty set X with a constant 0 and a binary operation ∗ satisfying the following conditions:

1. ((x ∗ y) ∗(x ∗z)) ∗(z ∗y) = 0
2. (x ∗ (x ∗y)) ∗ y = 0
3. x ∗ x = 0
4. 0∗x = 0
5. x ∗ y = 0& y ∗ x = 0 implies x = y, for any x, y, z ∈ X

Definition 2.2 A BCI-algebra (X, ∗,0) is a non empty set X with a constant 0 and a binary operation ∗ satisfying the following conditions:

1. ((x ∗ y) ∗(x ∗z)) ∗(z ∗y) = 0
2. (x ∗ (x ∗y)) ∗ y = 0
3. x ∗ x = 0
4. x ∗ y = 0 & y ∗ x = 0 implies x = y, for any x, y, z ∈ X

Definition 2.3 A BP- algebra (X, ∗,0) is a non empty set X with a constant 0 and a binary operation ∗ satisfying the following conditions:

1. x ∗ x = 0
2. x ∗ (x ∗ y) = y
3. (x∗ z)∗ (y∗z) = x∗ y), for any x, y, z ∈ X

Definition 2.4 Let S be a non-empty set. A mapping μ : S → [0, 1] is called a fuzzy subset of S.

Fuzzy BP-subalgebra

Definition 3.1 : A fuzzy subset μ of a BP-algebra (X,∗,0) is called a fuzzy BP subalgebra of X if, for all x,y ∈X the following condition is satisfied

μ(x∗ y) ≥ min { μ(x), μ(y) }

Example 3.2 Let X = {0,a, b, c } be a set with the following table:

∗	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then (X,∗, 0) is a BP – algebra

Defined as $\mu(x) = \begin{cases} .8 \; if x = 0 \\ .5 \; if x = b \\ .4 \; if x = a, c \end{cases}$

Then μ is a BP-subalgebra of X.

One can easily prove that:

Theorem 3.4 *Intersection of any two fuzzy BP-sub algebras of X is again a fuzzy BP- sub algebra.*

Definition 3.5 *Let μbe any fuzzy subset of a BP – algebra (X,∗, 0) and let t ∈[0, 1]*
*The set U(μ, t) = { x ∈**X** : μ(x) ≥ t }*
is called a level subset of μ of X.

Lemma 3.6 *Let (X,∗, 0) be a BP- sub algebra. Let* μ *be a fuzzy BP – subalgebra of X.*
Let α ∈[0, 1]. Then

1. *U(*μ*, α) is either ∅ or a BP- subalgebra of X*
2. μ(0) ≥ μ(x) *for all x ∈**X***

Proof:

For anyα ∈ [0, 1], assume that U(μ, α) is non-empty .
Let x, y ∈U(μ, α). Therefore μ*(x)*≥ *α*, μ*(y)* ≥ *α*
To show that U(μ, α) is a BP – subalgebra, we need to show x∗y∈U(μ, α).
That is, we need to show $\mu(x * y) \geq min$ {μ(x), μ(y)} $\geq min\,\{\alpha, \alpha\}$ $= \alpha$
Also, μ(0) = μ(x ∗ x) ≥min {μ(x), μ(x)}
= μ(x)
Since x∗ x = 0 ∀ x ∈ X
Thus μ(0) ≥ μ(x), ∀ x ∈ X

Lemma 3.7 *A fuzzy subset μ of a BP – subalgebra X is a fuzzy BP- subalgebra if and only if for all t ∈[0, 1], the level set of μ, U(μ, t) is either empty or a BP – subalgebra of X.*

Proof:

Assume that the level subset of μ in X,
U(μ, t) ≠ ∅
Then for any x, y ∈U(μ, t),
μ(x) ≥ t, μ(y) ≥ t
Now, μ(x∗ y) ≥ min { μ(x), μ(y) } ≥t
which implies x ∗ y∈U(μ, t) and hence U(μ, t) is a BP – subalgebra of X.
Conversely assume that U(μ, t) is a BP- subalgebra of X

Take t = min { μ(x), μ(y) } for any x, $y \in$ X

x, $y \in$ X *implies* x ∗ y∈U(μ, t)

Hence μ(x ∗ y) ≥ t = min { μ(x), μ(y) }, thus proving that μ is a fuzzy BP – subalgebra of X.

Lemma 3.8 *Any BP – subalgebra of a BP- algebra (X,* , 0) can be realized as a level subalgebra of some fuzzy BP-subalgebra of X*

Proof:

Let A be a BP – subalgebra of X and μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t \; if \, x \in A \\ 0 \quad otherwise \end{cases}$$

where t ∈ [0, 1]

If x, y ∈A then x∗ y ∈ A.

Therefore μ(x) = μ(y) = μ(x∗ y) = t and

μ(x∗ y) ≥ min { μ(x), μ(y) }

If both x, y ∉A

Then 0 = μ(x ∗ y) ≥ min { μ(x), μ(y) }

If at most one of x, y ∈ A, then also we have x ∗ y ∉A.

Hence at least one of μ(x) or μ(y) is equal to 0.

Therefore, μ(x∗ y) = 0 ≥ min{ μ(x), μ(y) }

This shows that A is a level subalgebra of X corresponding to the fuzzy BP- subalgebra μ of X.

From the above two lemmas we get the following theorem.

Theorem 3.9 *Let A be a subset of X. Then the characteristic function* χ_A *is a fuzzy BP- subalgebra of X if and only if A is a BP- subalgebra of X*

Theorem 3.10 Let μ be a fuzzy BP- subalgebra of (X,∗, 0) with finite image. If U(μ, s) = U(μ, t) for some s, t ∈ Im(μ), then s = t.

Proof:

Let μ be a fuzzy BP- subalgebra of X with finite image such that

U(μ, s) = U(μ, t) for some s, t ∈ Im(μ).

Now, μ is a fuzzy algebra of X shows that U(μ, s) is a BP-subalgebra.

Therefore, if x, y ∈U(μ, t) = U(μ, s) then μ(x) ≥ t and μ(y) ≥ t.

Also, x, y ∈U(μ, t) = U(μ, s) and U(μ, s) is a BP-subalgebra shows that x∗ y ∈U(μ, s).

This shows that

μ(x∗ y) ≥ min { μ(x), μ(y) } ≥ s.

Thus we have, μ(x∗ y) ≥ s as well as μ(x∗ y) ≥ t whenever x, y ∈U(μ, t) = U(μ, s).

Similarly, we can prove that, μ(x∗ y) ≥ s as well as μ(x∗ y) ≥ t whenever

x, y ∈U(μ, s) = U(μ, t).

This proves that s = t.

Lemma 3.11 *Let μ and λ be two fuzzy BP – sub algebras of X with identical family of level BP – sub algebras. If Im(μ) = {* $t_1, t_2, \ldots t_n$*} and Im(λ) = {*$s_1, s_2, \ldots, s_m$*} where F* $t_1 \geq t_2 \geq \ldots \geq t_n$ *and* $s_1 \geq s_2 \geq, \ldots, \geq s_m$ *Then*

1. *m = n*
2. *U(μ,* t_i*) = U(λ,* s_i*) for i = 1, 2, ….n*
3. *If μ(x) =* s_i *, then λ(x) =* s_i*, ∀x∈ X and i = 1, 2,….. n*

Proof:

Let μ and λ be two fuzzy BP – sub algebras of X with identical family of level BP – sub algebras *F*(μ) = *F*(λ).

Let *Im(μ) = {* $t_1, t_2, \ldots t_n$*} where*

$t_1 \geq t_2 \geq \ldots \geq t_n$ (1.1)

and Im(λ) = {$s_1, s_2, \ldots, s_m$*} where*

$s_1 \geq s_2 \geq, \ldots, \geq s_m$ (1.2)

(1.1) *implies* $U(\mu, t_1) \subseteq U(\mu, t_2) \subseteq \ldots \subseteq U(\mu, t_n) = X$ (1.3)

(1.2) implies $U(\lambda, s_1) \subseteq U(\lambda, s_2) \subseteq \ldots \subseteq U(\lambda, s_n) = X$ (1.4)

and $F(\mu) = \{U(\mu, t_i) : 1 \leq i \leq n\}$,

$F(\lambda) = \{U(\lambda, s_i) : 1 \leq j \leq m\}$

Assume m≠n.

Then, m ≥ n of n ≥ m.

Let m≥ n.

Then $U(\mu, t_i) = U(\lambda, s_i)$, *i = 1, 2,….. n.*

This shows that both t_i *and* s_i *Im*(μ).

For i> n we observe that $t_i \notin Im(\mu)$ and hence,

$U(\mu, t_i) \neq U(\lambda, s_i)$ *i =n+1, n+2, …m.*

Let n ≥ m. Then $U(\mu, t_i) = U(\lambda, s_i)$

i = 1, 2,….. m. This shows that both t_i *and* s_i *Im(λ)*. For j> m we observe that $s_j \notin Im(\mu)$ and hence,

$U(\mu, t_i) \neq U(\lambda, s_i)$ *i =m+1, m+2, …n.*

(1.3) and (1.4) implies $t_i \neq, s_i$,

∀i = 1, 2,….. n

Hence we can find some i such that $U(\mu, t_i) \neq U(\lambda, s_i)$.

This contradicts that F(μ) = F(λ).

Hence we conclude that m = n.

1. By part(1), we have proved that m = n. Since μ and λ have identical family of level sub algebras, we have

$U(\mu, t_i) = U(\lambda, s_i)$, *i = 1, 2,….. n.*

2. Follows from (1) and (2)

Let μ(x) = t_i, *implies* λ(x) = s_i,

for i = 1, 2, ….,n

Theorem 3.12 *Let μ and λ be two fuzzy sub algebras of X with identical family of level sub algebras. Then Im(μ) = Im(λ) implies μ = λ*

Proof Let μ and λ be two fuzzy sub algebras of X with identical family of level sub algebras.

Let *Im(μ) = Im(λ) = {*$s_1, s_2, \ldots, s_n$*}*

Where $s_1 \geq s_2 \geq, \ldots, \geq s_n$

By lemma [3.11] for any x∈ X, there exists s_i such that *μ(x) =* s_i = λ(x).

Thus *μ(x) =* λ(x) *∀x∈ X, proving that μ =* λ

Theorem 3.13 *Two level BP- sub algebras U(μ, s) and U(μ, t), (s<t) of a fuzzy BP- subalgebra μ are equal if and only if there is no x∈ X such that s≤* $\mu(x)$ *<*t.

Proof Let U(μ, s) and U(μ, t) be two level BP-sub algebras of fuzzy BP-subalgebra μ of X

Suppose that U(μ, s) = U(μ, t) for some *s<t.*

Suppose there is one x∈ X such that *s≤* $\mu(x)$ <t.

Then, $\mu(x) \geq$ s and $\mu(x) <$ t.

That is, x ∈U(μ, s) and x ∉U(μ, t).

This contradicts to U(μ, s) = U(μ, t).

Conversely, assume that there is no x∈ X such that $s \leq \mu(x) < t$.
Suppose, U(μ, s) ≠U(μ, t)
For, x∈U(μ, t) ⟹ $\mu(x) \geq t > s$
⟹ $\mu(x) > s$ ⟹x∈U(μ, s)
Since
U(μ, s) ≠ U(μ, t), choose U(μ, s) ⊄U(μ, t).
Hence there is an x∈U(μ, s) and
x ∉U(μ, t). ⟹μ(x) ≥ s and μ(x) < t.
Thus there exists an element x∈ X such that $s \leq \mu(x) < t$, thus contradicting our hypothesis.
Hence U(μ, s) = U(μ, t).

Definition 3.14 *Let λ and μ be the fuzzy set in a set X. the Cartesian product*
λx μ : X x X → [0, 1] is defined by
(λx μ)(x,y) = min {λ(x), μ(y)}∀x∈ X.

Theorem 3.15 *If μ_1 and μ_2 are fuzzy BP – sub algebras of X, then $\mu = \mu_1 \times \mu_2$ is a fuzzy BP – subalgebra of X x X.*

Proof.
For any (x_1, x_2) and (y_1, y_2) ∈ X x X, we have,
$\mu((x_1, x_2)*(y_1, y_2)) = \mu(x_1 * y_1, x_2 * y_2)$
$= (\mu_1 \times \mu_2)(x_1 * y_1, x_2 * y_2)$
$= \min\{\mu_1(x_1 * y_1), \mu_2(x_2 * y_2)\}$
$\geq \min\{\min(\mu_1(x_1), \mu_1(y_1)), \min(\mu_2(x_2), \mu_2(y_2))\}$
$= \min\{\min(\mu_1(x_1), (\mu_2(x_2)), \min(\mu_1(y_1)\mu_2(y_2))\}$
$= \min\{(\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2)\}$
$= \min\{\mu(x_1, x_2), \mu(y_1, y_2)\}$
Hence $\mu = \mu_1 \times \mu_2$ is a fuzzy BP – subalgebra of X x X.

Definition3.16 *Let $(X_1, *_1, o_1)$ and $(X_2, *_2, o_2)$ be BP-algebras. A mapping $f: X_1 \to X_2$ is called a homorphism if,*
$$f(x *_1 y) = f(x) *_2 f(y) \ \forall x, y \in X$$

Definition 3.17 *Let f be any function from the BP-algebra X_1 to the BP- algebra X_2. Let* μ be any fuzzy BP- subalgebra of X_1 satisfying supreimum property and σ be any fuzzy BP – subalgebra of X_2. The image of μ under f, denoted by f(μ), is fuzzy subset of X_2 defined by
$$(f(\mu(y)) = \begin{cases} Sup_{x \in f^{-1}(Y)} \mu(x) & if\, f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$
where y ∈X_2. The pre image of σ under f, symbolized by $f^{-1}(\sigma)$, is a fuzzy subset of X_1 defined by
$$(f^{-1}(\sigma))(x) = \sigma(f(x)) \ \forall x, \in X_1.$$

Lemma3.18 *Let$(X_1, *_1, o_1)$ and $(X_2, *_2, o_2)$ be two BP-algebras. Let $f: X_1 \to X_2$ be an epimorphism. If σ is fuzzy BP- subalgebra of X_2 then $f^{-1}(\sigma)$* is a fuzzy BP-subalgebra of X_1.
Alternatively, we have, epimorphic pre image of a fuzzy BP- subalgebra is a fuzzy BP- sub algebra.

Proof.
$(f^{-1}(\sigma))(x *_1 y) = \sigma(f(x *_1 y))$
$= \sigma(f(x) *_2 f(y))$ since f is an epimorphism
$\geq \min(\sigma(f(x), \sigma(f(y)))$ since σ is a fuzzy BP – sub algebra
$= \min((f^{-1}(\sigma))(x), f^{-1}(\sigma))(x)))$ $\forall x, y \in X$
Thus $f^{-1}(\sigma)$ is a fuzzy BP- subalgebra of X_1.

Lemma 3.19 *An epimorphic image of a fuzzy BP-subalgebra satisfying sup property is a fuzzy BP- sub algebra. That is, let $f: X_1 \to X_2$ be an epimorphism of BP- algebras. If* μ is a fuzzy BP – subalgebra of X_1 with sup property, then f(μ) is a fuzzy BP – subalgebra of X_2.

Proof:
Let *f(x), f(y)∈ f(X_1) and let $x_o \in f^{-1}(f(x))$, and $y_o \in f^{-1}(f(y))$, be such that*
$\mu(x_o) = Sup_{a \in f^{-1}(f(x))} \mu(a)$
$\mu(y_o) = Sup_{b \in f^{-1}(f(y))} \mu(b) (f(\mu)(x*y)$
$= Sup_{a \in f^{-1}(f(x*y))} \mu(a)$ if $f^{-1}(x*y) \neq \emptyset$.
Let *$A = f^{-1}(f(x))$, $B = f^{-1}(f(y))$, $C = f^{-1}(f(x).f(y))$*
$A * B = \{x \in X_1 : x = a * b : a \in A, b \in B\}$, $x \in A * B$
$f(x) = f(a * b) = f(a) * f(b)$, $x \in (f^{-1} f(a) * f(b))$ implies $A * B \subseteq C$.
Now,
$f(\mu)(f(a) *_2 f(b)) = \underset{a \in f^{-1}(f(a) *_2 f(b)))}{Sup} \mu(x)$
$= \underset{x \in C}{Sup}\, \mu(x) \geq \underset{x \in A*B}{Sup}\, \mu(x) \geq \underset{a \in A, b \in B}{Sup}\, \mu(a *_1 b)$
$\geq \underset{a \in A, b \in B}{Sup} \min(\mu(a), \mu(b))$
$= \min \underset{a \in A, b \in B}{Sup} (\mu(a), \mu(b))$
$= \min \left\{ \underset{a \in f^{-1}(f(x))}{Sup} \mu(a), \underset{b \in f^{-1}(f(x))}{Sup} \mu(b) \right\}$
$= \min\{f(\mu(a)), f(\mu(a))\}$
Thus an epimorphic imageof a fuzzy BP – subalgebra satisfying the sup property is a fuzzy BP – sub algebra.

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