

A STUDY ON GENERALIZED HERMITE POLYNOMIALS

K.S. DHOUNSI, YASMEEN

Abstract: In this paper, we obtain generating functions involving hyper geometric functions. Rodrigues type formula of Hermite polynomials which is closely related to generalized Hermite polynomials of Dattoli et. al. These results provide useful extensions of the well known results of classical Hermite polynomials $H_n(x)$.

Keywords: Generating functions, Hermite polynomials, Gould-Hopper polynomials, Rodrigues type formula, Dattoli et.al.

AMS(MOS): Subject Classification (2000): 33 Special Functions, 33C45, 33C80

Introduction: The Gould-Hopper polynomials $g_n^m(x, y)$ [2; p.512 (18)] see also [3; p.58 (6.2)] are generalization of classical Hermite polynomials $H_n(x)$ [6; p.187(2)]. The notation $H_n^{(m)}(x, y)$ for $g_n^{(m)}(x, y)$ was given by Dattoli et.al. and by Pathan, Yasmeen and Qureshi [4]. It is defined by [3; 6; p.76(6)]

$$g_n^m(x, y) = H_n^{(m)}(x, y) = \sum_{k=0}^{[n/m]} \frac{n! y^k x^{n-mk}}{k! (n - mk)!} \dots(1.1)$$

$$= x^n {}_mF_0 \left[\begin{matrix} \Delta(m; -n; \\ - \end{matrix}; y \left(\frac{-m}{x} \right)^m \right]. \dots (1.2)$$

where m is a positive integer and $\Delta(m; -n)$ abbreviates the array of m parameters,

$$\frac{-n}{m}, \frac{-n+1}{m}, \dots, \frac{-n+m-1}{m}; m \geq 1.$$

These polynomials reduce, when $m = 2$ and $y = -1$, to the classical Hermite polynomials. The equation (1.1) can be derived from the following generating relations [2; p.512 (19)]

$$\sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{t^n}{n!} = \exp(xt + yt^m) \dots (1.3)$$

and reduces to Hermite polynomials of two variables [2; p.511(11)]

$$H_n(x, y) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (2x)^{n-2k} y^k}{k! (n - 2k)!} \dots (1.4)$$

$$= H_n^{(2)}(2x, -y) \dots (1.5)$$

which can be derived from the generating relations [2; p.510(8)]

$$\sum_{n=0}^{\infty} \frac{H_n(x, y) t^n}{n!} = \exp(2xt - yt^2) \dots (1.6)$$

$$H_n(x, 1) = H_n(x) \dots (1.7)$$

In this paper we shall give some basic relations and properties involving the generalized Hermite polynomials $H_n(x, y)$ and then take up generating function and Rodrigues type formula for $H_n(x, y)$ and $H_n(x)$ are derived as special cases.

Generating Functions

Theorem - 1: Any values of parameters and variables leading to result which do not make sense are tacitly excluded then

$$\sum_{n=0}^{\infty} (c)_n H_n(x, y) \frac{t^n}{n!} = (1 - 2xt)^{-c} \times {}_2F_0 \left[\begin{matrix} \frac{c}{2}, \frac{c+1}{2}; \\ - \end{matrix}; \frac{-4yt^2}{(1 - 2xt)^2} \right]; |2xt| < 1 \dots(2.1)$$

Proof: Consider the following series

$$\sum_{n=0}^{\infty} (c)_n H_n(x, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(c)_n (-1)^k (2x)^{n-2k} y^k t^n}{k! (n - 2k)!}$$

Replacing n by $n+2k$, and using Legendre's duplication formula, we get

$$= \sum_{k=0}^{\infty} (1 - 2xt)^{-c-2k} \frac{\left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k (-4yt^2)^k}{k!} = (1 - 2xt)^{-c} {}_2F_0 \left[\begin{matrix} \frac{c}{2}, \frac{c+1}{2}; \\ - \end{matrix}; \frac{-4yt^2}{(1 - 2xt)^2} \right]$$

which is required result (2.1).

Theorem-2: Any values of parameters and variables leading to result which do not make sense are tacitly excluded, then

$$\sum_{n=0}^{\infty} {}_2F_0[-n, c; -; v] H_n(x, y) \frac{t^n}{n!} = \exp(2xt - yt^2) \times [1 + 2(x - yt)vt]^{-c} {}_2F_0 \left[\begin{matrix} \frac{c}{2}, \frac{c+1}{2}; \\ - \end{matrix}; \frac{-4y v^2 t^2}{(1 + 2xvt - 2yv t^2)^2} \right]$$

Proof: Consider the following series

$$S = \sum_{k=0}^{\infty} \frac{(c)_k H_k(x - yt, y) (-vt)^k}{k!} \dots(2.3)$$

Now using [4; p.452 (2.4)], we get

$$S = \exp(-2xt + yt^2) \times$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (c)_k v^k H_{n+k}(x, y) t^{n+k}}{k! n!}.$$

Replacing n by n-k and using

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}, 0 \leq k \leq n, \text{ we get}$$

S =

$$\exp(-2xt + yt^2) \sum_{n=0}^{\infty} {}_2F_0[-n, c; -; \nu] H_n(x, y) \frac{t^n}{n!}.$$

... (2.4)

Now again by (2.3) and using (2.1), we get
S =

$$[1 + 2v(x - yt)t]^{-c} {}_2F_0\left[\frac{c}{2}, \frac{c+1}{2}; -; \frac{-4yv^2 t^2}{(1 + 2xvt - 2yvt^2)^2}\right]$$

... (2.5)

Now equating the equation (2.4) and (2.5), we get required result (2.2).

Rodrigues type formula

Theorem-3

If $D = \frac{\partial}{\partial x}$, then $H_n(x, y) = 2^n \exp\left(-\frac{y}{4} D^2\right) x^n$

... (2.6)

Proof: Again by generating relations (1.6), we get

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} &= \exp(2xt) \sum_{n=0}^{\infty} \frac{(-yt^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{y}{4} D^2\right)^n \exp(2xt) \\ &= \exp\left(-\frac{y}{4} D^2\right) \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!} \end{aligned}$$

Equating the coefficient of t^n , we get required result (2.6).

Theorem-4 Use the fact that

References:

1. Brafman, F.; Some generating functions of Laguerre and Hermite polynomials. *Canad. J. Math.* 9, (1957), 180-187.
2. Dattoli, G., Chiccoli, C., Lorenzutta, S. Torre, A., and Maino, G.; Generalized Bessel Functions and Generalized Hermite polynomials, *Jour. Math. Anal. Appl.*, 178 (1993), 509-516.
3. Gould, H.W. and Hopper, A.T.; Operational formulas connected with two generalization of Hermite polynomials. *Duke Math. J.* 29, (1962), 51-63.
4. Pathan, M.A., Yasmeen and Qureshi, M.I.; Linear and bilinear generating function involving

$$\exp(2xt - yt^2) = \exp[2(xt) - y(xt)^2] \exp(yt^2(x^2 - 1))$$

... (2.7)

To prove

$$H_n(x, y) = \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{n! H_{n-2s}(1, y) x^{n-2s} y^s (x^2 - 1)^s}{s! (n - 2s)!}$$

... (2.8)

Proof: By (2.7) we can write

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} H_n(1, y) \frac{(xt)^n}{n!} \sum_{s=0}^{\infty} \frac{y^s t^{2s} (x^2 - 1)^s}{s!} \end{aligned}$$

Replacing n by n - 2s, in right hand side and equating the coefficient of t^n , we get required result.

Special Cases : I. For $y = 1$, then (2.1) reduced to [6; p.190 (1)], which is published by Brafman [1] and for $c = 1$ was given by Truesdell [8].

For $y = 1, v = y$, then (2.2) reduces to

$$\begin{aligned} \sum_{n=0}^{\infty} {}_2F_0[-n, c; -; y] \frac{H_n(x) t^n}{n!} &= \exp(2xt - t^2) \\ &\times [1 + 2y(x - t)]^{-c} {}_2F_0\left[\frac{c}{2}, \frac{c+1}{2}; -; \frac{-4y^2 t^2}{(1 + 2xyt - 2yt^2)^2}\right] \end{aligned}$$

This is a well known result [6; p.198] and obtained by Brafman [1] with contour integration as the main tool.

II. For $y = 1$, (2.6) reduce to [5; p.129(2)].

III. For $y = 1$, then (2.8) reduces to

$$H_n(x) = \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{n! H_{n-2s}(1) x^{n-2s} (x^2 - 1)^s}{s! (n - 2s)!}$$

This is a known result [7; p.133(3)]

Special cases I to III are known formulae of generating functions, Rodrigues type formula for classical Hermite polynomial $H_n(x)$.

Department of Mathematics, Maharshi Dayanand College, Sri Ganganagar - 335001, Rajasthan, India
Department of Mathematics, Government College, Kota-324010, Rajasthan, India