ALUTHGE TRANSFORMATION ON N - CLASS A(k)OPERATORS

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Abstract: In this paper, we study and discussed the concepts of polar decomposition and Aluthge transformation. Also we derived and characterized the Aluthge transformation on a new class of operators, N-class A(k) operators for fixed N > 0 and $x\epsilon H$.

Keywords: Class A(k) operators, N-Class A(k) operators, Aluthge transformation.

Introduction: Let H be a non zero complex Hilbert space and let B(H) denote the algebra of all bounded linear operators on H. An operator T is said to be class A if $|T^2| \geq |T|^2$where $|T| = (TT^*)^{\frac{1}{2}}$. For each k > 0, an operator T belongs to class A(k) if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$. An operator T is called absolute k-paranormal for k>0 $\||T|^k Tx\| \geq \|Tx\|^{k+1}$ for every unit vector $x\epsilon H$.

Furuta [4] introduced class A(k) and absolute–k–paranormal operators respectively. It has been proved that every log–hyponormal operator is class A(k), every class A(k) operator is absolute–k–paranormal operators and every paranormal operator is k–paranormal [4]. Yamazaki [7] proved that if T is an invertible class A(k) operator for k∈(0,1] then T ^{n}belongs to class$A(\frac{k}{n})$which is smaller class of operators than class A(k).

An operator T is said to be hyponormal if $T^*T \geq TT^*$ and also T is said to be semihyponormal if $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$ and semihyponormal was introduced by Xia [6]. It is known in Xia [6] that there exists an example which is semihyponormal but not hyponormal, that is, the class of semihyponormal operators properly contains the one of hyponormal operators. An operaor T on a Hilbert space H is said to be p–hyponormal if

$$(T^*T)^p(TT^*)^p forapositivenumberp.$$

The class of p–hyponormal has been defined as an extension of semihyponorma l and also it has been studied by many authors, mainly Aluthge [1], [2], Duggal [3] and Xia [6].

An operator means a bounded linear operator on a Hilbert space. It is well known that an operator T can be decomposed into T=U|T| where U is a partial isometry withN (U) = N (|T |), where N (X) denotes the kernel of an operator X and T = U |T | is saidto be the polar decomposition of an operator T if this kernel condition N(U) =N(|T|) is satisfied.

In this paper, we study the Aluthge transform $\tilde{T}$ (defined below) of an operator T on Hilbert space H has been studied extensively.

N-Class A(k) Operators: Let T be a bounded linear operator on a Hilbert space H. In [1], A. Aluthge introducedthe operator $\tilde{T}$ for an operator T with its polar decomposition T=U|T| =$|T^*|U$ and Takashi [5] introduced and defined $\tilde{T}$ and $\widetilde{T^*}$ as follows

$$\widetilde{T_{s,t}} = |T|^s U |T|^t$$

$$\widetilde{T_{s,t}}^* = (\widetilde{T_{s,t}})^* = |T|^t U^* |T|^s$$

Defintion 2.1. An operator $T\epsilon B(H)$ is N-class A(k)if $|T|^2 \leq N(T^*|T|^{2k}T)^{\frac{1}{k+1}}$ for a fixed N >0.

Theorem 2.2. Let $T\epsilon B(H)$. If T is N-class A(k),then $\|Tx\| \leq N\|(T^*(T^*T)^kT)\|^{\frac{1}{k+1}}\|x\|$.

Proof. Let T is N-class A(k) operator. Then

$|T|^2 \leq N(T^*|T|^{2k}T)^{\frac{1}{k+1}}$for a fixed N > 0.

$T^*T \leq N(T^*(T^*T)^kT)^{1/(k+1)}$

$\Leftrightarrow N(T^*(T^*T)^kT)^{1/(k+1)} - T^*T \geq 0,$

$\Leftrightarrow \langle \left(N(T^*(T^*T)^kT)^{\frac{1}{k+1}} - T^*T\right)x, x\rangle \geq 0,$

$\Leftrightarrow \langle \left(N(T^*(T^*T)^kT)^{\frac{1}{k+1}}\right)x, x\rangle \geq \langle T^*Tx, \quad x\rangle$

For all xϵ H,

Thus

$\|Tx\| \leq N\|(T^*(T^*T)^kT)\|^{\frac{1}{k+1}}\|x\|$.

Proposition 2.3. An operator $T\epsilon B(H)$. If T is N-class A(k) operator if $|T|^2 \leq N(T^*|T|^{2k}T)^{\frac{1}{k+1}}$ for a fixed N > 0

(i) If N = 1 then the operator is class A(k) operator

(ii) If N = 1 and k =1 then the operator is class A operator

(iii) If k = 1 then the operator is N class A operator.

Theorem 2.4. T is N-class A(k) operator for N > 0 if and only if $|T^*|^2 \leq N(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}$.

Proof. If T is N-class A(k) operator then

$|T|^2 \leq N(T^*|T|^{2k}T)^{\frac{1}{k+1}}$for a fixed N > 0.

$N(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}$

$= N\left[(TT^*)^{\frac{1}{2}}|T|^{2k}(TT^*)^{\frac{1}{2}}\right]^{\frac{1}{k+1}}$

$= N\left[(T^*)^{\frac{1}{2}}(T^*)^{\frac{1}{2}}|T|^{2k}(T)^{\frac{1}{2}}(T)^{\frac{1}{2}}\right]^{\frac{1}{k+1}}$

$= N[T^*|T|^{2k}T]^{\frac{1}{k+1}}$

$\geq U|T|^2U^*$

$= |T^*|^2$

Theorem 2.5. Let T=U|T|$\epsilon B(H)$ *be the polar decomposition of* T. *Then* T ϵ N-class A(k) operator for N > 0 if and only if $|T^*|^2 \leq N(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}$.

Proof. LetT=U|T|be the polar decomposition of T. ThenT*=U*|T*|is also thepolar decomposition of T*.

Suppose that T is a N-Class A(k) operator then,

$N(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}$

$= N\left[(TT^*)^{\frac{1}{2}}|T|^{2k}(TT^*)^{\frac{1}{2}}\right]^{\frac{1}{k+1}}$

$= N((U|T|U^*|T^*|)^{\frac{1}{2}}|T|^{2k}(U|T|U^*|T^*|)^{\frac{1}{2}})^{\frac{1}{k+1}}$

$= N((U|T||T|U^*)^{\frac{1}{2}}|T|^{2k}(U|T||T|U^*)^{\frac{1}{2}})^{\frac{1}{k+1}}$

$= N[|T^*||T|^{2k}|T|]^{\frac{1}{k+1}}$

$= N\left[(T)^{\frac{1}{2}}(T^*)^{\frac{1}{2}}|T|^{2k}(T)^{\frac{1}{2}}(T^*)^{\frac{1}{2}}\right]^{\frac{1}{k+1}}$

$= N\left[(T^*)^{\frac{1}{2}}(T^*)^{\frac{1}{2}}|T|^{2k}(T)^{\frac{1}{2}}(T)^{\frac{1}{2}}\right]^{\frac{1}{k+1}}$

$= N(T^*|T|^{2k}T)^{\frac{1}{k+1}}$

$\geq U|T|^2U^*$

$= |T^*|^2$

Therefore,

$N|T^*|^2 \leq N(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}$

Conversely, suppose that

$N(T^*|T|^{2k}T)^{\frac{1}{k+1}}$

$= N\ (U^*|T^*||T|^{2k}|T^*|U)^{\frac{1}{k+1}}$

$= \ U^*N(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}U$

$\geq U^*|T^*|^2\ U = \ |T|^2$

$N(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$

Hence T is N-class A(k) operators.

Theorem 2.6. Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$ the following assertion hold:

1. If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ then $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$,
2. If $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subset N(B)$ then $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.

Theorem 2.7. If T is N-class A(k) operator then T is $\frac{1}{k+1}$ hyponormal.

Proof. By assumption T is N-class A(k) operator. The following inequalities hold $N(T^*|T|^{2k}T)^{\frac{1}{k+1}} = N(|T|U^*|T|^{2k}U|T|)^{\frac{1}{k+1}} \geq |T|^2 \Leftrightarrow N(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2 - - - - - (1)$

By applying theorem 2.6.we obtain

$$|T|^{2k} \geq N(|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}} - - - - - (2)$$

$$|T|^2 \geq N(|T|^k|T^*|^2|T|^k)^{\frac{1}{k+1}} - - - - - - (3)$$

From (1) and (3) we get,

$N(|T|U^*|T|^{2k}U|T|)^{\frac{1}{k+1}} \geq |T|^2$

$\geq N(|T|^k|T^*|^2|T|^k)^{\frac{1}{k+1}} - - - - - - - - (4)$

$N(|T|U^*|T|^k|T|^kU|T|)^{\frac{1}{k+1}} \geq N(|T|^kTT^*|T|^k)^{\frac{1}{k+1}}$

$N(|T|U^*|T|^kS)^{\frac{1}{k+1}} \geq N(|T|^kU|T|U^*|T^*||T|^k)^{\frac{1}{k+1}}$

$N(S^*S)^{\frac{1}{k+1}} \geq N(SS^*)^{\frac{1}{k+1}}$

This shows that $S = \ |T|^kU|T|$ is $\frac{1}{k+1}$ hyponormal.

Theorem 2.8. If T is N-class A(k) operator then T^* is also N-class A(k) operator.

Proof. $N(T|T^*|^{2k}T^*)^{\frac{1}{k+1}} = N(TT^kT^{*k}T^*)^{\frac{1}{k+1}}$

$= \ N(T^{k+1}T^{*k+1})$

$= \ N(T^{*k+1}T^{k+1})$

$= \ N(T^*|T|^{2k}T)^{\frac{1}{k+1}}$

$\geq |T|^2$.

So that T^* is also N-class A(k) operator.

Theorem 2.9. If T is a N-class A(k) operator and S is an unitary operator such that TS = ST then C = TS is also N-class A(k) operator.

Proof.

$|C^2| \leq N(C^*|C|^{2k}C)^{\frac{1}{k+1}}$

$C^*C \leq N(C^*C^{*k}C^kC)^{\frac{1}{k+1}}$

$\leq NN\left(C^{*k+1}C^{k+1}\right)^{\frac{1}{k+1}}$

$\leq N\ (C^*C\ )$

$(TS)^*(TS) \leq N((TS)^*(TS))$

$\leq N(S^*T^*TS)$

$T^*T \leq N(T^*T)(since S is unitary)$

Hence, TS is N-class A(k) operator.

Theorem 2.10. Let T is N-class A(k) operator and S is a self adjoint operator on H. If $T^* = T$then

$$\|Tx\| \leq N\|(T^*|T|^{2k}T)\|^{\frac{1}{k+1}}\|x\|.$$

Proof. $N[T^*|T|^{2k}T]^{\frac{1}{k+1}} \geq |T|^2$

$N(T^*(T^*T)^kT)^{\frac{1}{k+1}} \geq (T^*T)$

$N[(T^*(T^*T)^kT)^*]^{\frac{1}{k+1}} \geq (T^*T)^*$

$N(T^*T^{*k}T^kT)^{\frac{1}{k+1}} \geq (T^*T)$

$N[T^*|T|^{2k}T]^{\frac{1}{k+1}} \geq |T|^2$

(i.e) $\|Tx\| \leq N\|(T^*|T|^{2k}T)\|^{\frac{1}{k+1}}\|x\|$

Therefore N-class A(k) operator is a self adjoint operator.

Theorem 2.11. Let $T = U|T|$ be the polar decomposition of N-class A(k) operator for N > o then $\widetilde{T_{s,t}} = |T|^sU|T|^t$ is $2(\frac{p+\min(s,t)}{s+t})$- N-class A(k) operator for s, t > o.

Proof. Let T be a N-class A(k) operator then

$|T|^2 \leq N(T^*|T|^{2k}T)^{\frac{1}{k+1}}$

$|T|^2 \leq T^*T$

$= (\widetilde{T_{s,t}}\ , \widetilde{T_{s,t}})^{(\frac{p+\min(s,t)}{s+t})}$

$= (|T|^tU^*|T|^s|T|^sU|T|^t)^{\frac{p+\min(s,t)}{s+t}}$

$= (|T|^tU^*|T|^{2s}U|T|^t)^{\frac{p+\min(s,t)}{s+t}}$

$= (\widetilde{T^*_{s,t}}\widetilde{T_{s,t}})^{(\frac{p+\min(s,t)}{s+t})}$

$= \left|\tilde{T}\right|^{2(\frac{p+\min(s,t)}{s+t})}$

$N(T^*|T|^{2k}T)^{\frac{1}{k+1}} = N(\widetilde{T^*_{s,t}}\widetilde{T^{*k}_{s,t}}\widetilde{T^k_{s,t}}\widetilde{T_{s,t}})^{\frac{1}{k+1}(\frac{p+\min(s,t)}{s+t})}$

$= N[(|T|^tU^*|T|^s)(|T|^tU^*|T|^s)^k(|T|^sU|T|^t)^k(|T|^sU|T|^t)]^{\frac{1}{k+1}\frac{p+\min(s,t)}{s+t}}$

$= N[(|T|^tU^*|T|^s)(|T|^sU|T|^t)]^{\frac{p+\min(s,t)}{s+t}}$

$= N(\widetilde{T^*_{s,t}}\widetilde{T_{s,t}})^{\frac{p+\min(s,t)}{s+t}}$

$= N|T|^{2(\frac{p+\min(s,t)}{s+t})}$

Hence we get,

$\widetilde{T_{s,t}} = |T|^s U|T|^t$is2($\frac{p+\min(s,t)}{s+t}$)- N-class A(k) operator for s, t > 0.

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