ALUTHGE TRANSFORMATION ON POWERS OF N-CLASS A_k OPERATORS

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Abstract: In this paper, we studied properties and conditions of powers of N-class A_k operator. Also we discussed polar decomposition of powers of N-class A_k operator and aluthge transformation on powers of N-class A_k operators for a fixed $N > 0$ and $x \in H$.

Keywords: Aluthge Transformation, Hyponormal Operators, N-Hyponormal Operators, N-Paranormal Operators, Polar Decompositions.

Introduction: Let H be a non zero complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on$H$. We mean a closed linear manifold of H by a subspace M of H and we mean a bounded linear transformation of H into itself by an operator T on$H$. A subspace M is invariant for T if$T(M) \subseteq M$. An operator T is called normal if$T^*T = TT^*$. An operator T is called quasi-normal if$T(T^*T) = (TT^*)$, it is called hypo normal if$T^*T \geq TT^*$. An operator T it is called quasi-hyponormal if $T^{*^2}T^2 = (T^*T)^2$. An operator $T \in B(H)$ is said to be $*-$ paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$for every$x \in H$. Many authors have been studied $*-$ paranormal operators and the following alterative definition is well known [1], [2], [3].

An operator $T \in B(H)$ is said to be $*-$paranormal if and only if $0 \leq T^{*^2}T^2 - 2\lambda TT^* + \lambda^2 I$ for allλ, T is $*-$paranormal if and only if $\lambda\|T^*\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \lambda^2\|x\|^2)$ for every$x \in H$, for all$\lambda > 0$. An operator is called M^*- paranormal operator if $\|T^*x\|^2 \leq M\|T^2x\|$for every unit vector$x \in H$. The class Q if it is defined and studied by Duggal et al [4]. An operator T is said to be class Q if $\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \lambda^2\|x\|^2)$ for every$x \in H$. Arora and Thukral [3] studied a new class of operators, namely quasi $*-$paranormal.

They also showed many example to prove the inclusion relation, normal $\subset k-$hyponormal $\subset k^*-$paranormal $\subset$ normaloid. Several extensions, of paranormal operators have being consider till now, for example class$A(k)$, absolute $k-$paranormal class, $k-$paranormal operators.

Panayappan and radhamani [6] introduced and characterized a new class of operators named as quasi -paranormal operators. Also introduced a new class of operators namely $p-*-$paranormal and absolute $k-*-$paranormal operators and showed that every hyponormal operator is an $k-*-$paranormal operator and for each $k > 0$, every class$A(k^*)$ operator is an absolute $k-*-$paranormal operator. An operator T is said to be quasi $-*-$paranormal operator if $\|T^*Tx\|^2 \leq \|T^3x\|\|Tx\|$ for every$x \in H$. An operator T is said to be class$A$if $|T^2| \geq |T|^2$. An operator $T \in B(H)$ is said to be normaloid if$r(T) = \|T\|$, where $r(T)$ denotes the spectral radius of T.In this paper, defined a new class of operators is namely, powers of N-classA_k and study some basic properties of this class. Then we study the aluthge transformation on powers of N - classA_k operator on H.

2. Powers of N- class A_k operators: In this section we defined powers of N-classA_k operator and consider some basic properties. Generalization of concept of paranormal, hyponormal, N–paranormal, quasi – hyponormal, N – Hyponormal and N^*-paranormal are studied in Braha et al. For an operator$T \in B(H)$, we say that N – quasi-hyponormal, if$\|T^*Tx\|^2 \leq N\|T^2x\|$for fixed$N > 0$, T is $N-$paranormal if $\|Tx\|^2 \leq N\|T^2x\|$ for fixed$N > 0$, T is N^*-paranormal if $\|T^*x\|^2 \leq N\|T^2x\|$ for fixed $N > 0$ and for all unit vector for fixed$x \in H$. An operator $T \in B(H)$is said to be class$A$ if$|T^2| \geq |T|^2$. An operator T is called class$A_k$if $|T|^2 \leq (|T^{k+1}|)^{\frac{2}{k+1}}$ and absolute – $k-$paranormal if $\||T|^kTx\| \geq \|Tx\|^{k+1}$ for every $x \in H$. It has been proved that every log-hypo normal operator is class$A(k)$, every class$A(k)$ operator is absolute–$k-$paranormal and every paranormal operator is $k-$paranormal. Yamazaki proved that if T is an invertible class$A(k)$ operator for $k \in (0,1]$ then T^n belongs to class$A\left(\frac{k}{n}\right)$ which is a smaller class of operator than class$A(k)$.

Definition 2.1: An operator $T \in B(H)$ is powers of N-classA_k if $|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$ for a fixed $N > 0$ and $0 < p \leq 1$.

Proposition 2.2: If $T \in B(H)$is powers of N-class$A_k$operator if $|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$ for a fixed $N > 0$ and $0 < p \leq 1$.

(i) If p = 1 then the operator is N – class$A(k)$ operator.

(ii) If p = 1 and k = 1 then the operator is N- class A operator.

(iii) If N = 1 and p = 1 then the operator is class $A(k)$ operator.

(iv) If N = 1, p = 1 and k = 1 then the operator is class A operator.

Theorem 2.3 An operator $T \in B(H)$is powers of N-classA_k if $\|T^px\| \leq N\||T^{k+1}|\|^{\frac{2p}{k+1}}\|x\|$.

Theorem 2.4 Let $T \in B(H)$ is powers of N- classA_k operator then

(i) $\lambda T$is powers of N-classA_k operator.

(ii) The translate$T-\lambda$ need not be powers of N-classA_k operator.

Proof. From the definition of powers of N- classA_k operator

$|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$

$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$

$(\lambda T^*\lambda T)^p \leq N((\lambda T^*\lambda T)^{k+1})^{\frac{p}{k+1}}$

$(\lambda^*)^p(\lambda)^p(T^*T)^p \leq N(\lambda^*)^p(\lambda)^p((T^*T)^{k+1})^{\frac{p}{k+1}}$

$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$

$|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$

Therefore $\lambda T$is powers of N-classA_k operator.

The operator $T = U^* - 2$ where U is the unilateral shift since$2 \notin \sigma(U^*)$. Therefore $T-\lambda$ need not be powers of N-classA_k operator.

Theorem 2.5: If $T \in B(H)$ is powers of N- classA_k operator then there exists a unique self ad joint operator T^* then $\|T^p x\| \leq N\||T^{k+1}|\|^{\frac{2p}{k+1}}\|x\|$.

Proof. From the definition of powers of N- classA_k operator

$|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$

$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$

$((T^*T)^*)^p \leq N(((T^*T)^*)^{k+1})^{\frac{p}{k+1}}$

$(T^*(T^*)^*)^p \leq N((T^*(T^*)^*)^{k+1})^{\frac{p}{k+1}}$

$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$

$\|T^p x\| \leq N\||T^{k+1}|\|^{\frac{2p}{k+1}}\|x\|$.

Therefore T self adjoint operator.

Theorem 2.6: Let$T = U|T| \in B(H)$ be the polar decomposition of$T$. Then T is a powers of N-class A_k operator if and only if $\||T|^p x\| \leq N\||T|^{k+1}x\|^{\frac{p}{k+1}}$.

Proof. By assumption T is a powers of N-class A_k operator

$|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$

$(T^*T)^p = N((T^*T)^{k+1})^{\frac{p}{k+1}}$

$((T^*T)^p x, x) = (N((T^*T)^{k+1})^{\frac{p}{k+1}}x, x)$

$(U^*|T^*|^p U|T|^p x, x) = (N(U^*|T^*|^{k+1}U|T|^{k+1})^{\frac{p}{k+1}}x, x)$

$\leq N\big((U^*|T^*|^{k+1}U|T|^{k+1})x, x\big)^{\frac{p}{k+1}}$

$\||T|^p x\| \leq N\||T|^{k+1}x\|^{\frac{p}{k+1}}$.

Converse also similar proof.

Theorem 2.7: If $T \in B(H)$ is powers of N- classA_k operator such that$(T^*T)^p = (TT^*)^p$,Then $T^*$is also powers of N- classA_k operator.

Proof. From the definition of powers of N- classA_k operator

$|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$

$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$

$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$

$(TT^*)^p \leq N((TT^*)^{k+1})^{\frac{p}{k+1}}$

Form above two equations

$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$

Therefore

$\|T^p x\| \leq N\||T^{k+1}|\|^{\frac{2p}{k+1}}\|x\|$.

Therefore $T^*$is also powers of N- classA_k operator.

Theorem 2.8: If $T \in B(H)$ is powers of N- classA_k operator and S is an unitary operator such that $TS = ST$, then $C = TS$ is also powers of N- classA_k operator.

Proof. From the definition of powers of N- classA_k operator

$|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$

$(C^*C)^p \leq N((C^*C)^{k+1})^{\frac{p}{k+1}}$

$((TS)^*(TS))^p \leq N(((TS)^*(TS))^{k+1})^{\frac{p}{k+1}}$

$(S^*T^*ST)^p \leq N((S^*T^*ST)^{k+1})^{\frac{p}{k+1}}$

$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$

$|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$

Therefore $T$is powers of N- classA_k operator.

Theorem 2.9: If $T \in B(H)$ is powers of N- classA_k operator for some $p > 0$ then T is p −paranormal operator.

Proof. For a given unit vector $x \in H$ and$T = U|T|$, we have,

$$\begin{aligned}\|T^{2p}x\| &= \||T|^p\|^2\\ &= (U^*|T^*|^{2p}Ux, x)\\ &\leq \Big(U^*(|T^*|^{k+1})^{\frac{2p}{k+1}}Ux, x\Big)\\ &= \Big((U^*|T^*|^p|T|^{2p}|T^*|^pU)^{\frac{1}{2}}x, x\Big)\\ &= \big((U^*|T^*|^p|T|^{2p}|T^*|^pU)x, x\big)^{\frac{1}{2}}\\ &= \big((|T|^pU^*|T|^{2p}U|T|^p)x, x\big)^{\frac{1}{2}}\\ &= \||T|^pU|T|^p\|\end{aligned}$$

Therefore powers of N- class A_k operator is p- paranormal operator.

3.Aluthge Transformation On powers of N-class A_k Operators: An operator T can be decomposed into $T = U|T|$ where U is partial isometry and $|T|$ is the square root of T^*T with $N(U) = N(|T|)$ and this U is the kernel condition $N(U) = N(|T|)$ uniquely determines U and $|T|$ in the polar decomposition of T. In this section we consider new properties as an extension of p −hypo normal operator using the generalized aluthge transformation. For an operator $T = U|T|$ defined $\tilde{T}$ as follows:

$$\tilde{T}_{s,t} = |T|^s U |T|^t$$

For $s, t > 0$ which is called generalized aluthge transformation of$T$. In this section we will study powers of N- classA_k operators using their generalized aluthge transform.

Theorem 3.1: Let $T = U|T|$ be the polar decomposition of powers of N- classA_k operators for $0 < p \leq 1$ then

$$\tilde{T}_{s,t} = |T|^s U|T|^t$$

Is $\frac{p+\min(s,t)}{s+t}$ powers of N- classA_k operator for$s, t > 0$ that$max(s,t) \geq p$.

Proof. From the definition of powers of N- classA_k operator

$$|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$$

$$|T|^{2p} = (T^*T)^p$$

$$= \left(\widetilde{T^*}_{s,t}\tilde{T}_{s,t}\right)^{p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= (|T|^t U^*|T|^s|T|^s U|T|^t)^{p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= (|T|^t U^*|T|^{2s} U|T|^t)^{p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= \left(\left|\tilde{T}_{s,t}\right|\right)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= N\left(|T^{k+1}|^{\frac{2p}{k+1}}\right)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= N\left(((T^*T)^{k+1})^{\frac{p}{k+1}}\right)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= N\left(((\widetilde{T^*}_{s,t}\tilde{T}_{s,t})^{k+1})^{\frac{p}{k+1}}\right)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= N(|T|^t U^*|T|^s|T|^s U|T|^t)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= N(|T|^t U^*|T|^{2s} U|T|^t)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$= N\left(\left|\tilde{T}_{s,t}\right|\right)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

$$\left(\left|\tilde{T}_{s,t}\right|\right)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}} \leq N\left(\left|\tilde{T}_{s,t}\right|\right)^{2p\left\{\frac{p+\min(s,t)}{s+t}\right\}}$$

Therefore

$$\tilde{T}_{s,t} = |T|^s U|T|^t$$

Is $\frac{p+\min(s,t)}{s+t}$ powers of N- classA_k operator.

References:

1. T. Ando, "Operator with a norm conditions". Acta. Math. Szeged , 33(1986): 19-178.
2. Thomas Monoth, Babu Anto P, Visual Cryptography Schemes With Perfect Reconstruction; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 706-709
3. S. C. Arora and J. K. Thukral. "On a class of Operators". Glasnimathematicki, 21(1986): 381-386.
4. S. C. Arora and J. K. Thukral. " M^* −Paranormal Operators". Glasnimathematicki, 21(1987): 123-129.
5. S. C. Arora and J. K. Thukral. "Quasi− ∗ −Paranormal Operators". Invariant subspace and Allied spaces, Narosa publication, (1990): 79-86.
6. Vaddiparthi Yogeswara , Biswajit Rath , S.V.G.Reddy, A Study of FS-Functions and Properties of Images; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 254-265
7. B. P. Duggal, and C. S. Kubrusly and N. Levan " Contraction and Centred Composition Operators". Proc. Royal Irish Acad, 90(1990):165-172.
8. S. Panayappan and A. Radhamani " on class of quasiparanormal Operators" . Int. Math. Analysis, 2(15),(2008): 174-745.
9. *Dr. Aparna Borkar , Dr. M.S. Gupta,* Behavior of Antenna input Spectral Impedance for Reactance ; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 141-144

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