SIGNAL PROCESSING EMBEDDED WITH FOURIER TRANSFORM

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Abstract: In this paper, we have discussed about signals, systems and transforms. Signal can be considered as functions. Audio signal is a one dimensional function of time whereas an image is a two dimensional function that varies in space. Systems manipulate signals, i.e. they can be considered as a map from functions to functions. Transforms make signal processing easier by changing the domain in which the underline signal is represented.

Key words: Signal, Transforms, Image, Domain

Introduction: Signal processing is a facilitate technology that includes the fundamental theory, algorithms and implementations of processing information contained in many different physical, symbolic or abstract formats broadly designated as signals. Analog signal processing is for signals that have not been digitized as in legacy radio, telephone radar and television systems. Whereas digital signal processing is the processing of digitized discrete-time sampled signals. This processing is execute by digital circuit such as field programmable gate arrays or specialized digital signal processors (DSP chips). Discrete Fourier Transform (DFT) convert the sampled (reduction of a continuous signal to a discrete signal) function from its original time domain to the frequency domain.

2.0 Signals: Signals can be functions of time or space. For processing signals it is advantageous to represent them in a frequency domain. Several operations such as filtering out unnecessary frequencies can be performed efficiently in the frequency domain. Generally low frequency components represent the overall space of the signal and high frequency components are caused due to noise or presence of edges in an image. To remove noise high frequency components are reduced. To sharpen an image high frequency components are emphasized. Thus it is easier to carry many operations in the frequency domain.

2.1 **Time domain to frequency domain:** Fourier discovered that any periodic signal can be represented as a combination of sinusoids of different amplitudes, frequencies and phases. Any periodic signal $S(t)$ with period T_0 (fundamental frequency $f_0 = \frac{1}{T_0}$) can be represented as follows

$$S(t) = \sum_{n=0}^{\infty} a_n \sin(2\pi n f_0 t + \phi) ----(1)$$

A phase shifted sine wave can be expressed in terms of sine and cosine functions due to the following identity.

$$Sin(A+\phi) = SinA\cos\phi + CosASin\phi$$

Thus we can express $S(t)$ as sum of sines and cosines of different amplitudes and frequencies. Thus the equation (1) reduces to

$$S(t) = \sum_{n=0}^{\infty} A_n \sin(2\pi n f_0 t) + B_n Cos(2\pi n f_0 t) \quad --(2)$$

Which also can be written as

$$S(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(2\pi n f\ t) + B_n Cos(2\pi n f t) \quad --(3)$$

If the function has non zero value for f_0, it indicates the presence of cosine components. A_0 represents the average value of the function over a time period. This is because the average value of sines and cosines over a time period is zero. Using the identities $\cos t = \frac{e^{it} + e^{-it}}{2}$ and $\sin t = \frac{e^{it} - e^{-it}}{2i}$ and let $c_n = \frac{A_n - iB_n}{2}$, $c_{-n} = \frac{A_n + iB_n}{2}$

Thus the equation (3) can be represented by means of a single complex exponential as

$$S(t) = \sum_{n=0}^{\infty} C_n e^{j2\pi n f_0 t} + C_{-n} e^{j2\pi n f_0 t}$$

Which can be written as $S(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$

The above formula known as Fourier decomposition of S(t) into complex exponentials. This decomposition is a way of transforming the signal from the time domain to frequency domain.

2.2 Vector space Model: The set of signals form a vector space. A vector space is a collection of vectors that is closed under linear combination. i.e. if two vectors $\vec{u}, \vec{v}$ are in the space then the vector $a\vec{u} + b\vec{v}$ will be also in the space for the scalars a and b. A familiar example of vector space is the Euclidean space R^3. Any set of objects with the operations addition and multiplication which are closed under linear combination forms a vector space. The length or norm of vector $\vec{v}$ is given by $\|v\| = \left(\sum_{i=0}^{n} v^2{}_i \right)^{\frac{1}{2}}$. The distance between two vectors v_1 and v_2 is the norm of the difference vector $v_1 - v_2$. The angle between two vectors is closely related to the inner product $< v_1, v_2 >$ due to the following relation.

$$\cos \vartheta = \frac{< v_1, v_2 >}{\|v_1\| * \|v_2\|}$$

Two vectors are said to be orthogonal if their inner product is zero. The concept of length, distance and angle can be defined on any vector space. For instance, in the space of functions, the integral $\int f(x).g(x)$ generalizes the dot product in which the summation is replaced by integration. The functions $f(x)$ and $g\{x)$ are said to be orthogonal on [a,b] if

$$\int_a^b f(x).g(x)dx = 0 .$$

For example the set of functions $\{e^{j2\pi nft_0}\}, n = 0,1,2,......$ are orthogonal.

A basis is a minimal set of vectors that spans the entire space. If the elements in basis are orthogonal and have unit length then they form orthogonal basis as formed by (i, j, k) in the Euclidean space. A point in the Euclidean space given by $u = xi + yj + zk$. The values x, y and z gives the magnitude of projection on these basis. To compute the magnitude of projection along *i*, we compute the inner product of **u** and *i*. Thus $x = ui$.

A function S(t) can be viewed as a point in the space of functions. The Fourier decomposition of the function is given by $S(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nf_0 t}$

Can be viewed as projection of the function onto orthogonal basis given by the complex exponential functions $e^{j2n\pi f_0 t}$. Finding the magnitude of projection is equivalent to finding the Fourier coefficients. To compute the magnitude of the projection along $e^{j2n\pi f_0 t}$, we compute the inner product of S(t) and $e^{j2n\pi f_0 t}$. Using the definition of inner product for functions , we get the Fourier coefficients as $C_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} S(t) e^{-j2n\pi f_0 t}\, dt$

3.0 **Systems:** Systems maps signals to signals. To characterize a system we need to find its response for each and every input function, which is infeasible. Fortunately the systems which are linear and time invariant (LTI) can be characterized if its response for a small set of inputs is known. In a linear system the principle of superposition holds. If x_1 and x_2 produces responses y_1 and y_2 respectively, then the input $ax_1 + bx_2$ Produces the response $ay_1 + by_2$. A system is said to be time invariant if a delay in input by t seconds causes the output to be delayed by t seconds.

3.1 Impulse Response: The impulse signal is a function with narrow width and infinitely large amplitude (Integration of rectangular signal shown in the below fig.)

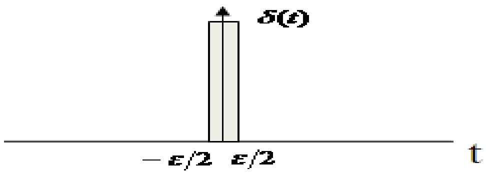
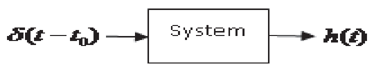


Figure: Impulse signal

$$\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[u(t + \varepsilon/2) - u(t - \varepsilon/2) \right]$$

Unit impulse signal has unit amplitude and exists only at instant of time. It is denoted by $\delta(t)$ and is referred as dirac delta function. The shifted version of impulse signal has a spike at time t = t_0 and is denoted by $\delta(t - t_0)$. This shifted version of this unit impulse signal is considered for studying the characteristics of the system.

The unit impulse signal is used to analyze the newly designed system. The response of the system in an initial condition for the given input of unit impulse signal is termed as impulse response denoted by $h(t)$.



This impulse response $h(t)$ is used to characterize the behavior of the system under designed specifications. It is generally used as system identification tool box. Then the same system can be represented by its own impulse response $h(t)$

Impulse response can be used to analyze the natural response of the designed system in all frequencies. It is mainly useful in convolution and deconvolution of data processing. Convolution process implies in the retrieval of x(t) from y(t).

LTI systems have a remarkable property that they can be completely characterized by its response to the impulse function. The input signal is decomposed into an infinite number of scaled and shifted impulse functions. Each of these impulses produces a scaled and shifted version of the impulse response in the output signal. The final output signal is then equal to the combined effect. i.e. the sum of all the individual responses. Thus if the response to the impulse function is known the response of the input signal can be calculated. Thus the output is the convolution of the input with the impulse response of the system.

3.2 Convolution: Convolution is a mechanism by which an input signal is shaped by another signal to produce an output signal. We introduce convolution by considering the problem of smoothing a signal. Later we show that the output of a system is the convolution of the input with the impulse response.

Consider a transformation that smoothes the signal by averaging adjacent signals $b_n = \frac{a_n + a_{n-1}}{2}$. This kinds of transformations are used for reducing effects of noise. The transformation can also be described in terms of polynomial multiplication. If we represent the input sequence $a_0, a_1, a_2, a_3, ...$ by the polynomial $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$, the coefficient of b_n, in the product $B(x) = A(x)\frac{1}{2}(1+x)$ is given by $b_n = \frac{a_n + a_{n-1}}{2}$ which is the output sequence. The polynomial $P(x) = \frac{1}{2}(1+x)$ defines how the input signal is manipulated (convolved) to produce the output. Convolution and polynomial multiplication are closely related.

For the product $B(x) = A(x) * P(x)$, the coefficient of the convolved signal b_n is given by $\sum_{i=0}^{n} a_i * P_{n-i}$

In the continuous domain the summation is replaced by integration and the convolution of two functions $f(x)$ and $g(x)$ becomes

$$C(u) = \int_0^u f(x) * g(u-x)dx$$

We can now show that the output of a system can be viewed as a convolution of the input signal with the impulse response of the system. Consider a system whose output is $h(t)$ when given the input $\delta(t)$. Any input signal can be considered as summation of scaled and shifted impulse functions. i.e. for a continuous signal S(t) can be viewed as $\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$. Since the response of $\delta(t)$ is $h(t)$, the response for s(t) is $\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$.

Thus the convolution of the input signal with the impulse response gives the output signal.

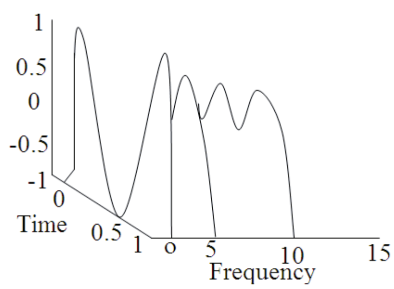
4.0Transforms and their applications in signal processing: Transform make it easier for systems to manipulate signals. Any linear transformation can be described as a projection onto a new set of basic functions.

We have seen the Fourier series for a periodic signal having frequency f_0 is given by

$S(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$ and that c_n is given by

$$C_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} S(t)e^{-j2n\pi f_0 t}\, dt$$

The following figure plots the frequency components and the magnitude of C_n



Note that here the amplitudes are function of frequencies defined at discrete points.

As the frequency f_0 decreases, these amplitudes become more closely spaced. For a non periodic signal the time period $T_0 \to \infty$ and thus $f_0 \to 0$. The separation between successive harmonics becomes small and the amplitudes become a continuous function of frequency. Thus nf_0

becomes the continuous variable and hence the formula for C_n becomes $C_f = \frac{1}{T_0}\int_{-\infty}^{\infty} S(t)e^{-j2\pi ft}\,dt ------(4)$ The above transformation takes a function of time as input and produces a function of frequency, i.e. $S(t) \rightarrow C_f$. This is called the Fourier transform. The transform makes it easier to perform many operations that are different in the time domain. The Fourier transform of the convolution of two functions $S_1(t)$ and $S_2(t)$ is equal to the product of their Fourier transforms. Intuitively multiplication emphasizes those frequencies that are present in both the signals. Thus for producing the convolution of two functions, it is easier to transform them into frequency domain, multiply them and transform back to the time domain.

4.1 Discrete Fourier Transform: While processing signals on a computer, variables are discrete and continuous Fourier transform is approximated by Discrete Fourier transform. We sample a continuous function $S(t)$ at intervals of Δ, generating a sequence of sampled values $S_k = S(k\Delta)$ collecting a total of N samples ($\frac{1}{\Delta}$ is the sampling frequency, which according to Nyquist theorem must be at least twice the frequency of the highest component present in the wave).

In the continuous Fourier transform, we have the values of the signal defined at each point of time and generate a function of frequency that is defined for all values. In DFT, we have the values of the signal at discrete points $t_k = k\Delta$ represented as S_k and generate a function of frequency that are valid at discrete frequencies C_n. In the discrete case the equation (4) will becomes

$$C_n = \sum_{k=0}^{N-1} S(k)e^{-2\pi ikn/N} -------(5)$$

The equation $x^N = 1$ is of degree N and thus has N roots given by $\varpi_N^{\ n}$ (n=0,1,....,N and $\varpi = e^{\frac{2\pi i}{N}}$

The equation (4) can also be written as

$$C_n = \sum_{k=0}^{N-1} S(k)\varpi^{kn}{}_N -------(6)$$

The DFT can be viewed as a transformation of N point signal into N Fourier coefficients or as a matrix that maps an N-dimensional vector to N-dimensional vector.

DFT can be used to speed up polynomial multiplication. We have already seen that polynomial multiplication is related to convolution and it is easy to convolve signals in frequency domain. We use these facts to speedup multiplication.

Considering the coefficients of a polynomial (of degree N) as our signal, we get N Fourier coefficients. (According to the equation (5), these correspond to the values of the polynomial evaluated at the complex roots of unity). These values give the point value representation of the polynomial. To multiply two polynomials $f(x)$ and $g(x)$, we evaluate the polynomials at roots of unity, multiply the corresponding values to get the point value representation of $f(x) * g(x)$ from which we can construct the polynomial $f(x) * g(x)$.

4.2 Fast Fourier Transform: The fast Fourier transform is a computationally fast method to compute the DFT. Computing DFT directly for N points requires $O(N^2)$ complex multiplication. FFT exploits the structure of DFT to reduce the complexity. Rewriting the equation (6) we get

$$C_n = \sum_{k=0}^{N-1} S(k)\varpi^{kn}{}_N \Rightarrow C_n = \sum_{k\,is\,even} S(k)\varpi^{kn}{}_N + \sum_{k\,is\,odd} S(k)\varpi^{kn}{}_N \Rightarrow C_n = \sum_{j=0}^{(N/2)-1} S(2j)\varpi_N^{\ 2jn} + \sum_{j=0}^{(N/2)-1} S(2j+1)\varpi_N^{\ (2j+1)n}$$

Using the relation $\varpi_N^{\ 2} = \varpi_{\frac{N}{2}}$, we get

$$C_n = \sum_{j=0}^{(N/2)-1} S(2j)\varpi_{N/2}^{\ jn} + \varpi_N^{\ n}\sum_{j=0}^{(N/2)-1} S(2j+1)\varpi_{N/2}^{\ jn}$$

Thus we have reduced the problem of computing N point DFT for $S(k)$ to finding two N/2 point DFTs for $S(j)$ and $S(2j+1)$. This is a divide and conquer approach where we have reduced the problem of size N to two sub problems of size N/2. It has complexity $O(N \log N)$

4.3 Discrete Cosine Transform: If the function is an even function, then the Fourier transform consists of only cosine components and transform becomes discrete cosine transform. Thus DCT can be defined in terms of DFT by imposing symmetry. DCT is used for compression in JPEG images. JPEG compression takes two account that the human eye, cannot detect high frequency components. Using a process of quantization, we retain all low frequency components. Whereas the high frequency components are retained only if they have large coefficients. On the resulting coefficients run length encoding and Huffman compression is applied.

Conclusion: We have seen how a signal can be viewed as a function in the vector space model of functions. Systems are used to manipulate signals and a special class of system called LTI system can be easily characterized by its impulse function. Transforms make it is easier to manipulate signals. Transforms can be viewed as change of basis in the vector space model. Transforms have a number of applications including filtering, convolution, pattern recognition, solving partial differential equations, multiplication of polynomials and large numbers, Analysis of time series etc.

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