GENERALIZATION OF URYSHON'S LEMMA VIA WEAK FORM OF OPEN SETS

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Abstract: The purpose of this paper is to investigate some basic properties of $(1,2)^*\beta$-σ_i-continuous functions and $\beta^*_{1,2}$-normal spaces. Also we prove a famous lemma called Uryshon's Lemma by using $\beta^*_{1,2}$-normal spaces.

Keywords: Bitopological spaces, $(1,2)^*\beta$ -open sets, $(1,2)^*\beta$-σ_i-continuous functions, $\beta^*_{1,2}$-normal spaces.

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Introduction: In 1963, Kelly [4] initiated the study of the bitopological space which is to be a set X equipped with two topologies $\tau_1$and $\tau_2$on X. In bitopological spaces, Lellis Thivagar and Ravi [5,6] introduced $(1,2)^*\alpha$-open sets by defining a new class of open sets namely $\tau_{1,2}$-open sets and also developed the weak forms open sets called $(1,2)^*$semi-open sets and $(1,2)^*$pre-open sets. Lellis Thivagar and Athisaya Ponmani [7] introduce $(1,2)^*\beta$-open sets and $(1,2)^*\beta$-σ_i-continuous functions and investigate certain properties on it. This paper is to focus on some more properties of $(1,2)^*\beta$-σ_i-continuous functions between bitopological spaces and $\beta^*_{1,2}$-normal spaces. Finally, we highlight the proof of Uryshon's Lemma by using $(1,2)^*\beta$-σ_i-continuous functions and $\beta^*_{1,2}$-normal spaces.

2. Preliminaries: In this section we recollect some properties of basic concepts which are useful in the sequel.

Definition 2.1.[4] A non-empty set X together with two topologies $\tau_1$and $\tau_2$is called a bitopological spaces and is denoted by (X, τ_1, τ_2).

Definition 2.2.[5, 6]A subset S of a bitopological space (X, τ_1, τ_2) is called $\tau_{1,2}$-open if and only if $S = A\cup B$, where $A \in \tau_1$ and $B \in \tau_2$.

The family of all $\tau_{1,2}$-open sets is denoted by $\tau_{1,2}O(X)$. Note that $\tau_{1,2}O(X)$ need not necessarily form a topology and $\tau_1O(X), \tau_2O(X) \subseteq\tau_{1,2}O(X)$.

Remark 2.3 [5, 6] Let A be a subset of a bitopological space (X, τ_1, τ_2).
Then

(i) $\tau_{1,2}\text{-int}(A) = \cup\{G: G \subseteq A \text{ and } G \text{ is } \tau_{1,2}\text{- open}\}$

(ii) $\tau_{1,2}\text{-cl}(A) = \cap\{F: A \subseteq F \text{ and } F \text{ is } \tau_{1,2}\text{-closed}\}$.

Definition 2.4 A subset A of a bitopological space (X, τ_1, τ_2) is called

(i) $(1,2)^*\alpha$-open [6] if $A \subseteq\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A)))$

(ii) $(1,2)^*$semi-open [6] if $A \subseteq\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A))$

(iii) $(1,2)^*$pre-open [6] if $A\subseteq\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A))$ and

(iv) $(1,2)^*\beta$-open [7] if $A \subseteq\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A)))$.

The complement of $(1,2)^*\alpha$-open (resp. $(1,2)^*$semi-open, $(1,2)^*$pre-open and $(1,2)^*\beta$-open) sets are called $(1,2)^*\alpha$-closed (resp.$(1,2)^*$semi-closed,$(1,2)^*$pre-closed and $(1,2)^*\beta$-closed) sets. The family of all $(1,2)^*\alpha$-open (resp.$(1,2)^*$semi-open, $(1,2)^*$pre-open and $(1,2)^*\beta$-open) sets is denoted by $(1,2)^*\alpha O(X)$ (resp. $(1,2)^*SO(X)$, $(1,2)^*$
$PO(X)$ and $(1,2)^*\beta O(X)$).

Definition 2.5 [7] Let (X, τ_1, τ_2) be a i_α bitopological space and $S \subseteq X$. Then $(1,2)^*\beta$ closure of A is defined as $(1,2)^*\beta\ cl(A) = \cap\{F: A \subseteq F \text{ and } F \text{ is } (1,2)^*\beta\text{-closed}\}$and $(1,2)^*\beta$ interior of A is defined as $(1,2)^*\beta\ int(A) = \cup\{G: G \subseteq A \text{ and } G \text{ is } (1,2)^*\beta\text{-open}\}$.

Definition 2.6: [4] A bitopological space (X, τ_1, τ_2) is called pairwise normal if for every pair of disjoint τ_1-closed set F_1 and τ_2-closed set F_2 there exists disjoint τ_1-open set U and τ_2-open set V such that $F_1 \subseteq V$ and $F_2\subseteq U$.

Definition 2.7 [4] Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A ma $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(1,2)^*\beta$-σ_i-continuous if $f^{-1}(V)$ is $(1,2)^*\beta$-open, for every σ_i-open set V of Y, i= 1,2.

Definition 2.8 [10] A dyadic number is a number s that can be expressed as $s = m/2^n$, where m = 1, 3, 5, ... , 2^n -1 and $n \in N \cup\{0\}$. The set of dyadic numbers in [0,1], denoted by P, are certainly countable.

Remark 2.9 [10] The set of dyadic numbers in I is dense in I.

3. $\beta^*_{1,2}$-normal Spaces and

Uryshon's Lemma: In this section we introduce and establish the properties of a new type of space called $\beta^*_{1,2}$-normal space and discuss certain properties of $(1,2)^*\beta$-σ_i-continuous functions. Using these we prove the famous lemma namely Uryshon's Lemma.

Theorem 3.1 Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces and let $f: X\to Y$. Then the following statements are equivalent.

(i) f is $(1,2)^*\beta$-σ_i-continuous.

(ii) If $\boldsymbol{B_1}$and $\boldsymbol{B_2}$ are basis for σ_1 and σ_2 respectively, then $f^{-1}(B_i)$ is $(1,2)^*\beta$- open in X for each $B_i\in\boldsymbol{B_i}$ and i= 1,2.

(iii) If $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ are subbasis for σ_1 andσ_2 respectively, then $f^{-1}(S_i)$ is $(1,2)^*\beta$- open in X for each $S_i\in\boldsymbol{S_i}$ and i= 1,2.

Proof: (i) ⇒(ii): Since each basis element B_i of $\boldsymbol{B_i}$ is an element of $\sigma_iO(X)$ and f is $(1,2)^*\beta$-σ_i-continuous, $f^{-1}(B_i)$ is $(1,2)^*\beta$-open in X.

(ii) ⇒(i): Let $V_i$be a σ_i-open set in (Y, σ_1, σ_2). Then there is a collection $\{B_{i\alpha} : \alpha \in\Lambda \}$ of members of $\boldsymbol{B_i}$ for

σ_i such that $V_i = \cup\{B_{i\alpha} : \alpha \in \Lambda\}$ and $f^{-1}(V_i) = \cup \{f^{-1}(B_{i\alpha}) : \alpha \in \Lambda\}$. Therefore $f^{-1}(V_i)$ is $(1,2)^*\beta$-open in X.

(i) ⇒(iii): Since each subbasis element S_i of $\boldsymbol{S_i}$ is an element of $\sigma_i O(X)$ and f is $(1,2)^*\beta$-σ_i-continuous, $f^{-1}(\boldsymbol{S_i})$ is $(1,2)^*\beta$-open in X.

(iii) ⇒ (i): Since the family of all finite intersections of subbasis elements is a basis, then by part (ii), we get the result.

Definition 3.2 A bitopological space (X, τ_1, τ_2) is called $\beta^*_{1,2}$-normal if for every pair of disjoint $(1,2)^*\beta$-closed set F_1 and F_2 there are disjoint $(1,2)^*\beta$-open sets U and V such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Remark 3.3 The $\beta^*_{1,2}$-normality and pairwise normality are independent to each other, which shown in the following examples.

Example 3.4 Let X = {a, b, c, d}, τ_1 = {ϕ, X, {a}, {a, b, c}}, τ_2 = {ϕ, X, {b},{a, b},{a, b, d}}. Then $\tau_{1,2}$ O(X) = {ϕ, X,{a}, {b}, {a, b}, {a, c}, {a, b, c}, {a, b, d}} and $(1,2)^*\beta$ O(X) = {ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, d}, {a, b, c}, {a, c, d}, {a, b, d}}. Clearly (X, τ_1, τ_2) is $\beta^*_{1,2}$-normal but not pairwise normal.

Example 3.5 Let X = {a, b, c}, τ_1 = {ϕ, X, {a}} and τ_2 = {ϕ, X, {a, b}}. Then $\tau_{1,2}$ O(X) = {ϕ, X,{a},{a, b}} and $(1,2)^*\beta$ O(X) = {ϕ, X,{a},{a, b},{a, c}}. Clearly (X, τ_1, τ_2) is not $\beta^*_{1,2}$-normal but it is pairwise normal.

Theorem 3.6 A bitopological space (X, τ_1, τ_2) is $\beta^*_{1,2}$-normal if and only if for each $(1,2)^*\beta$-closed subset A of X and each $(1,2)^*\beta$-open subset U of X such that $A \subseteq U$, there is a $(1,2)^*\beta$-open subset V of X such that $A \subseteq V \subseteq (1,2)^*\beta\ cl(V) \subseteq U$.

Proof: Suppose A is $(1,2)^*\beta$-closed and U is $(1,2)^*\beta$-open such that $A \subseteq U$. Then X–U is $(1,2)^*\beta$-closed and $A \cap (X-U) = \phi$. Since (X, τ_1, τ_2) is $\beta^*_{1,2}$-normal, then there are disjoint $(1,2)^*\beta$-open sets V and W such that $A \subseteq V$ and $X-U \subseteq W$. Since $V \cap W = \phi$, $(1,2)^*\beta\ cl(V) \subseteq X-W \subseteq U$. Now suppose A and B are two disjoint $(1,2)^*\beta$-closed sets, then there is a $(1,2)^*\beta$-open set V such that $A \subseteq V \subseteq (1,2)^*\beta\ cl(V) \subseteq X-B$. Then clearly V and X – $[(1,2)^*\beta\ cl(V)]$ are two disjoint $(1,2)^*\beta$-open sets such that $A \subseteq V$ and $B \subseteq X - [(1,2)^*\beta\ cl(V)]$.

Theorem 3.7 Let (X, τ_1, τ_2) be a bitopological space and let P be a dense subset of [0,1]. Suppose that for each $j \in P$, there is a $(1,2)^*\beta$-open set U_j in X such that : (1) if s < t then $(1,2)^*\beta\ cl(U_s) \subseteq U_t$ and (2) $X = \cup_{j\in P} U_j$. Define f : X→ [0,1] by f(x) = glb{ $j \in P : x \in U_j$ }for each $x \in X$. Then f is $(1,2)^*\beta$-σ_i-continuous function.

Proof: Let S_I be the subbasis for the topologies σ_i on [0,1], i= 1,2 and that consists of all sets of the form [0, a) and (a,1], where $a \in (0,1)$. By Theorem 3.1, it is sufficient to show that both $f^{-1}([0, a))$ = { $x \in X$: f(x) <a} and $f^{-1}((a,1])$ = { $x \in X$: f(x) > a} are $(1,2)^*\beta$-open sets in X. To show that $f^{-1}([0, a)) = \cup \{ U_t : t \in P$ and t < a }. Let $y \in f^{-1}([0, a))$.Then $0 \le f(y) < a$. Since P is dense in [0,1], there is a $t_y \in P$ such that $f(y) < t_y < a$. That is glb{ $t \in P : y \in U_t$} < t_y< a. Hence $y \in U_{ty}$, where t_y< a and so $y \in \cup \{ U_t : t < a\}$. For the other inclusion, if $y \in \cup\{ U_t : t < a \}$, then $y \in U_{ty}$ for some $t_y \in P$ with t_y< a. This implies f(y) = glb{$t \in P : y \in U_t$} < t_y< a and also $y \in f^{-1}([0, a))$. Therefore $f^{-1}([0, a)) = \cup \{ U_t : t \in P$ and t < a } is the union of $(1,2)^*\beta$-open sets and hence it is $(1,2)^*\beta$-open. To show that $f^{-1}((a,1])$ is $(1,2)^*\beta$-open, we have to show $X - f^{-1}((a,1]) = \{x \in X : f(x) \le a \}$ is $(1,2)^*\beta$-closed. To show $\{ x \in X : f(x) \le a \} = \cap \{ (1,2)^*\beta\ cl(U_t) : t \in P$ and a < t }. Let $y \in \{ x \in X : f(x) \le a \}$and let $t \in P$ such that a < t. Then there is a $s \in P$ such that s < t and $y \in U_s$. Since $U_s \subseteq U_t$, $y \in U_t \subseteq (1,2)^*\beta\ cl(U_t)$ and hence$\{ x \in X : f(x) \le a \} \subseteq \cap\{(1,2)^*\beta\ cl(U_t) : t \in P$ and a < t }. For the reverse inclusion, let $y \in \cap \{(1,2)^*\beta\ cl(U_t) : t \in P$ and a < t } and let $\epsilon > 0$. Since P is dense in [0,1], there is $s_1 \in P$ such that $a < s_1 < a+\epsilon$. Now $y \in \{(1,2)^*\beta\ cl(U_{s_1})$. Again since P is dense in [0,1], there is $s_2 \in P$ such that $s_1 < s_2 < a+\epsilon$. Since $\{(1,2)^*\beta\ cl(U_{s_1}) \subseteq U_{s_2}$, $y \in U_{s_2}$. Therefore $f(y) \le s_2$.Since $s_2 < a+\epsilon$, $f(y) < a+\epsilon$.Then since ϵ is arbitrarily positive number, $f(y) \le a$. Thus $\{x \in X : f(x) \le a\} \supseteq \cap\{(1,2)^*\beta\ cl(U_t) : t \in P$ and a < t}. This completes the proof.

Theorem 3.8 (Uryshon's Lemma): A bitopological space (X, τ_1, τ_2) is $\beta^*_{1,2}$-normal if and only if for each pair A, B of disjoint $(1,2)^*\beta$-closed subsets of X there is a $(1,2)^*\beta$-σ_i-continuous function f : X → [0,1] such that f(x) = 0 for all $x \in A$ and f(x) = 1 for all $x \in B$.

Proof: Suppose (X, τ_1, τ_2) is $\beta^*_{1,2}$-normal and let A and B be disjoint $(1,2)^*\beta$-closed subsets of X. Since X–B is $(1,2)^*\beta$-open and $A \subseteq X-B$. By Theorem 3.6, there is a $(1,2)^*\beta$-open set $U_{1/2}$ such that $A \subseteq U_{1/2} \subseteq (1,2)^*\beta\ cl(U_{1/2}) \subseteq X-B$. We continue to use Theorem 3.6. Since $A \subseteq U_{1/2}$, there is a $(1,2)^*\beta$-open set $U_{1/4}$ such that $A \subseteq U_{1/4} \subseteq (1,2)^*\beta\ cl(U_{1/4}) \subseteq U_{1/2}$. Since $(1,2)^*\beta\ cl(U_{1/2}) \subseteq X-B$, there is a $(1,2)^*\beta$-open set $U_{3/4}$ such that $(1,2)^*\beta\ cl(U_{1/2}) \subseteq U_{3/4} \subseteq (1,2)^*\beta\ cl(U_{3/4}) \subseteq X-B$. Thus we have the following: $A \subseteq U_{1/4} \subseteq (1,2)^*\beta\ cl(U_{1/4}) \subseteq U_{1/2} \subseteq (1,2)^*\beta\ cl(U_{1/2}) \subseteq U_{3/4} \subseteq (1,2)^*\beta\ cl(U_{3/4}) \subseteq X-B$. Then we continue this process, we obtain two families of sets $\{U_j\}$ and $\{(1,2)^*\beta\ cl(U_j)\}$, where $j = m/2^n \in P$, P is the set of dyadic numbers. Let $U_1 = X$ so that each element of X will be in some U_j. Then we have $U_s \subseteq U_t \subseteq (1,2)^*\beta\ cl(U_s) \subseteq (1,2)^*\beta\ cl(U_t) \subseteq (1,2)^*\beta\ cl(U_u)$ for s < t < u, and $(1,2)^*\beta\ cl(U_t) \subseteq U_u$ for t < u. Now define f : X → [0,1] by f(x) = glb { $j \in P \cup \{1\} : x \in U_j$ }. By Theorem 3.7, f is $(1,2)^*\beta$-σ_i-continuous. If $x \in A$, then $x \in U_j$ for all $j \in P \cup \{1\}$ and hence f(x) = 0 for all $x \in A$. If $x \in B$, then $x \notin U_j$ for all $j \in P$ but $x \in U_1$ and so f(x) = 1. Now suppose A and B are two disjoint $(1,2)^*\beta$-closed subsets of X. Then there is a $(1,2)^*\beta$-σ_i-continuous function f : X → [0,1] such that f(x) = 0 for all $x \in A$ and f(x) = 1 for all $x \in B$. Hence $f^{-1}([0,1/2)) = \{ x \in X : f(x) < 1/2\}$ and $f^{-1}((1/2,1]) = \{ x \in X : f(x) > 1/2\}$ are disjoint $(1,2)^*\beta$-open sets such that $A \subseteq f^{-1}([0,1/2))$ and $B \subseteq f^{-1}((1/2,1])$.

Theorem 3.9 (General form Uryshon's Lemma): A bitopological space (X, τ_1, τ_2) is $\beta^*_{1,2}$-normal if and only if for each pair A, B of disjoint $(1,2)^*\beta$-closed subsets of X there is a $(1,2)^*\beta$-σ_i-continuous function f : X →[a, b] such that f(x) = a for all x ∈A and f(x) = b for all x ∈B.

Proof: Suppose (X, τ_1, τ_2) is $\beta^*_{1,2}$-normal and let A, B of disjoint $(1,2)^*\beta$-closed subsets of X. By Uryshon's Lemma, there is a $(1,2)^*\beta$-σ_i-continuous function f_1 : X → [0,1] such that $f_1(x)$ = 0 for all x ∈A and $f_1(x)$ = 1 for all x ∈B. If we take f(x) = (b–a) $f_1(x)$ + a, then f is $(1,2)^*\beta$-σ_i-continuous function. Also if x ∈A, then f(x) = a and if x ∈B, then f(x) =b. Now suppose A and B be disjoint $(1,2)^*\beta$-closed subsets of X and there is a $(1,2)^*\beta$-σ_i-continuous function f : X → [a, b] such that f(x) = a for all x ∈A and f(x) = b for all x ∈B. If we take $f_1(x)$ = (f(x) – a) /(b–a), then f_1 is $(1,2)^*\beta$-σ_i-continuous. Also if x ∈A, then $f_1(x)$ = 0 and if x ∈B, then $f_1(x)$ = 1. By Uryshon's Lemma, (X, τ_1, τ_2) is $\beta^*_{1,2}$-normal.

Conclusion: In this paper, we have discussed some more properties of $(1,2)^*\beta$-σ_i-continuous functions. Also we introduced and established the properties of $(1,2)^*\beta$-normal spaces. In future, we can insist these things into many research fields such as rough set topology, fuzzy set topology, digital topology, etc.

References:

1. Abd. El-Monsef.M.E, El Deeb. S.N and Mahmoud.R.A.: β-open sets and β- continuous mappings, Bull. Fac. Sci. Assiut Univ., 12(1983), 77-90.
2. V. Kaladevi, V.Manimozhi, on Characteristic Polynomial of Product of incidence; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 691-693
3. Arya.S.P and Bhamini.M.P.: Some generalizations of pairwise Uryshon spaces, Indian J. Pure and Applied. Math., 18(12) (1987), 1088-1093.
4. Dorsett.C.: Semi –normal spaces, Kyungpook Math J.,25(1985),173-180.
5. *Wasefer Zaman, Partha Ghosh,Kalyankumar Datta, Pramatha Nath Basu,* A Framework to incorporate Domain Knowledge Modeling ; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 474-478
6. Kelly.J.C.:Bitopological spaces, Proc. Londan Math.Soc(3),13(1963),71-89.
7. Lellis Thivagar.M.: Generalization of pairwise α-continuous functions, Pure and Applied Mathematicka Sciences , 28(1991) , 55-63.
8. *Kavita B. Bajpai,* Common Fixed Point theorem for Six Mappings in Partial *Anushri A.Aserkar, Manjusha P.Gandhi,* ; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 433-438
9. Lellis Thivagar.M and Ravi.O.: A bitopological (1,2)*-semi generalized continuous mappings,Bull of Malaysian. Math. Soc., 29(1)(2005).
10. Lellis Thivagar.M and Athisaya Ponmani, Note on some new bitopological separation axioms,Proc. of. the National conference in Pure and Applied Mathematics, (2005) , 28-32.
11. A.Praveenprakash ,M.P.Kannan ,J.Kirubakaran, A Study on the Causes for Child Trafficking; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 336-338
12. Maheswari.S.N, Prasad.R.: Some new separation axioms, Ann. Soc. Sci. Bruxelles 89, (1975),395-402
13. Maki.H; Devi.R; Balachanran.K, Generalized α-closed sets in Topology, Bull.Fukuoka Univ.Ed.Part III, 42, (1993),13-21.
14. Patty. C.W.: Foundations of Topology, PWS-KENT Publishing Company, Boston, (1992).
15. Reilly. I.L.: On bitopological separation properties, Nanta Mathematica, V(1972), 14-25.

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