SYMMETRIC *n*-DERIVATIONS ON PRIME AND SEMIPRIME COMMUTATIVE RINGS

DR. DHANANJAYA REDDY

Abstract: Let n ≥2 be a fixed positive integer and let R be a non-commutative n!-torsion free semi prime ring. Suppose that there exists a symmetric n-derivation $\Phi : \mathcal{R}^n \longrightarrow \mathcal{R}$ such that the trace of Φ is centralizing on R. Implies the trace is commuting on R. If R is an n!-torsion free prime ring and $\Phi \neq 0$ under the same condition. Implies $\mathcal{R}$is commutative.

Keywords: associative ring, additive map, commutative ring, semi prime ring, symmetric derivation.

Introduction And Preliminaries: Consider $\mathcal{R}$ represents an associative ring and Z its center. Let x; y; z ∈ R. Take the notation [y,x] for the commutate yx - xy with identities [xy; z] = [x; z]y + x[y; z] and [x,yz] = [x,y]z+y[x, z]. R is semi prime if aRa= 0 ⇒a = 0 and R is prime if aRb = 0 ⇒a = 0 or b = 0. A map f:R ⟶ R is said to be commuting on R if [f(x); x] = 0 ∀x ∈R. A map f:R ↦ R is centralizing on R if [f(x); x] ∈ Z is fulfilled ∀x ∈R. An additive map D:R ⟶ R is called a derivation if the Leibniz rule D(xy) = D(x)y + xD(y) holds ∀x, y ∈ R. Let n ≥2 be a fixed positive integer and R^n= R ×R × .... ×R. A map $\Phi : \mathcal{R}^n \longrightarrow \mathcal{R}$is said to be symmetric if the equation $\Phi(x_1, x_2 \dots x_n) = \Phi(x\pi_{(1)}, x\pi_{(2)} \dots x\pi_{(n)})$ holds $\forall x_i \in R$ and for every permutation $\{\pi(1), \pi(2), \dots \pi(n)\}$.

Let n ≥2 be a fixed positive integer. An n-additive map $\Phi : \mathcal{R}^n \longrightarrow \mathcal{R}$will be called an n-derivation if the relations

$\Phi(x_1x_1', x_2, \dots x_n) = \Phi(x_1, x_2, \dots \boldsymbol{x_n})x_1' + x_1\Phi(x_1', x_2, \dots x_n)$

$\Phi(x_1, x_2x_2' \dots x_n) = \Phi(x_1, x_2, \dots \boldsymbol{x_n})x_2' + x_2\Phi(x_1, x_2', \dots x_n)$

....

$\Phi(x_1, x_2 \dots x_nx_n') = \Phi(x_1, x_2, \dots x_n)x_n' + x_n\Phi(x_1, x_2, \dots x_n')$ are valid $\forall x_i, x_i' \in \mathcal{R}$.

Φ Is a symmetric bi-derivation on non-commutative 2-torsion free prime rings, M. Bresar [1, Theorem 3.5] proved that Φ = 0.

Let n ≥2 be a fixed positive integer. If R is commutative, then a mapΦ : $\mathcal{R}^n \longrightarrow \mathcal{R}$ defined by $(x_1, x_2, \dots x_n) \longmapsto D(x_1)\, D(x_2) \dots D(x_n)$ $\forall x_i \in \mathcal{R}$, i = 1, 2... n is a symmetric n-derivation, where D is a derivation on$\mathcal{R}$.

Let $\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} / a, b \in \mathbb{C} \right\}$;

Where ℂ is a Complex field. Clearly R is a non-commutative ring under matrix addition and matrix multiplication. And a map $\Phi : \mathcal{R}^n \longrightarrow \mathcal{R}$is a symmetric *n*-derivation.

Let n ≥2 be a fixed positive integer and let a map φ: R ⟶ R defined by $\varphi(x) = \Phi(x_1, x_2, \dots x_n)$∀ x∈R, where Φ : R^n⇢R is a symmetric map, be the trace of Φ It is obvious that, in case when φ: R ⟶ R is a symmetric map which is also n-additive, the trace φ of Φ satisfies the relation.

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k \text{ , } \forall \text{ x, y} \in R$$

A study on the theory of centralizing commuting maps on prime rings was initiated by the classical result of E.C. Posner [5] which states that the existence of a nonzero centralizing derivation on a prime ring R whichis commutative. The study of centralizing commuting maps, J. Vukman [6, 7] investigated symmetric bi-derivations on prime and semi prime rings. Also obtained the similar results to Posner's and Vukman's ones for permuting 3-derivations on prime and semi prime rings [3]. This paper is to apply the result due to E.C. Posner [5, Theorem 2] to symmetric *n*-derivations.

2. Some Results:

Precede the proof of the results by two well known lemmas.

Lemma 2.1 : Let R be a prime ring. Let$\mathcal{D}$: $\mathcal{R} \longrightarrow \mathcal{R}$ be a derivation and a∈ $\mathcal{R}$. If a$\mathcal{D}$(x) = 0 holds ∀x ∈ $\mathcal{R}$ then we have either a = 0 *or* $\mathcal{D}$= 0 ([4]).

Lemma 2.2: Let n be a fixed positive integer and let R be a n!-torsion free ring. Suppose that $y_1, y_2, \dots y_n \in \mathcal{R}$ which satisfy $\lambda y_1 + \lambda^2 y_2 + \dots + \lambda^n y_n$ for λ=1,2,...n. ⇒ y_i =0, ∀ i([2]).

Theorem 2.3: Let n ≥2 be a fixed positive integer and let $\mathcal{R}$be a non-commutative n!-torsion free prime ring. Suppose that there exists a symmetric n-derivationΦ: $\mathcal{R}^n \longrightarrow \mathcal{R}$ such that the trace φ of Φ is commuting on$\mathcal{R}$. Then we have Φ = 0.

Proof: Suppose that $[\delta(x), x] = 0$ ∀ x ∈ $\mathcal{R}$ (1)

Let $\lambda (1 \leq \lambda \leq n)$ be any integer.Substituting x+λy for x in (1) we have

$$\lambda\{[\delta(x), y] + C_1^n[h_1(x;y), x]\} + \lambda^2\{C_1^n[h_1(x;y), y] + C_2^n[h_2(x;y).x]\} + \dots + \lambda^n\{[\delta(y), x] + C_{n-1}^n[h_{n-1}(x;y), y]\} = 0 \;\forall x, y \in \mathcal{R} \quad (2)$$

From Lemma 2.2 and (2), we get,

$[\delta(x), y] + n[h_1(x;y), x] = 0$ ∀ x, y ∈ $\mathcal{R}$ (3)

Put xy=y in (3) we have

$[\delta(x), xy] + n[h_1(x; xy), x] = 0$

⇒$x\{[\delta(x), y] + n[h_1(x;y), x]\} + n\delta(x)[y, x] = 0$

⇒$n\delta(x)[y, x] = 0 = (x)[y, x]$, ∀ x, y ∈ $\boldsymbol{\mathcal{R}}$ (4)

⇒ $\delta(x)$ = 0 (From (4) and Lemma 2.1),∀ x ∈ $\mathcal{R}$ (x≠ Z)

Let $x \in \boldsymbol{\mathcal{R}} (x = Z)$and $y \in \mathcal{R}$ $(y \neq Z) \Rightarrow \boldsymbol{y + \lambda x \notin Z}$

∴ $\delta(y + \lambda x) = 0$

$$\Rightarrow \delta(y) + \lambda^n \delta(x) + \sum_{k=0}^{n-1} \lambda^k \binom{n}{k} h_k(y; x) = 0$$

$\sum_{k=0}^{n-1} \lambda^k \binom{n}{k} h_k(y;x) + \lambda^n \delta(x) = 0$

$\forall x, y \in \mathcal{R}$ **(5)**

$\Rightarrow \boldsymbol{\delta(x)} = 0$ (By applying (5) in Lemma2.2) $\forall\, x \in \mathcal{R}$ $(x \in Z)$

Now, let P_k (x) = Φ (x,x,…k times ,x_{k+1} , x_{k+2} ,… x_n) for k=1,2,…n and for x, $x_i \in R$,

i= k+1, k+2…n

Let $\mu \in Z$ such that $(1 \le \mu \le n-1)$

$\therefore \delta(\mu x + x_n) = 0$

$\Rightarrow P_n(\mu x + x_n) = 0$

$\Rightarrow \mu^n \delta(x) + \delta(x_n) + \sum_{k=1}^{n-1} \mu^k \binom{n}{k} P_k(x) = 0$

$\Rightarrow \sum_{k=1}^{n-1} \mu^k \binom{n}{k} P_k(x) = 0$ true $\forall x, x_n \in R$ (6)

∴From Lemma 2.1 and (6) we have

$C_{n-1}^{n} P_{n-1}(x) = 0 = P_{n-1} \forall\, x \in R$ (7)

Let ν $(1 \le \nu \le n-1)$ be any integer.

From (7), $P_{n-1}(\nu x + x_{n-1}) = 0$

$\Rightarrow \nu^{n-1} P_{n-1}(x) + P_{n-1}(x_{n-1}) + \sum_{k=1}^{n-1} \nu^k \binom{n}{k} P_k(x) = 0$ $,\forall x, x_{n-1} \in R$

$\Rightarrow \sum_{k=1}^{n-1} \nu^k \binom{n}{k} P_k(x) = 0 \quad \forall x \in R$ (8)

Now, $C_{n-1}^{n} P_{n-2}$ (x) = 0 = P_{n-2} (x) ,∀x∈ R (By (8) and Lemma 2.1)

By proceed above we get finally $C_1^n P_1$ (x) = 0 = P_1 (x) , ∀x∈ R

$\Rightarrow \Phi(x_1, x_2, \ldots x_n) = 0 \quad \forall x_i \in R$

Hence the Proof.

Lemma 2.4: Let n be a fixed positive integer and let R be a n!-torsion free ring. Let y_1 ,y_2 , …, $y_n \in$ R satisfy $\lambda y_1 + \lambda^2 y_2 + \ldots + \lambda^n y_n \in \mathcal{Z} \forall \lambda$ = 1,2,…n. Then $y_i \in \mathcal{Z} \forall$ i

Theorem 2.5: Let n ≥2 be a fixed positive integer and let R be a non-commutative n!-torsion free semi prime ring. Suppose that there exists a symmetric n-derivation $\Phi: \mathcal{R}^n \longrightarrow \mathcal{R}$ such that the trace φ of Φis centralizing on R. Then φ is commute on R.

Proof:

Assume that $[\varphi(x), x] \in \mathcal{Z}$, ∀x∈ $\mathcal{R}$ (9)

Let λ (1≤ $\lambda \le$ n) be any positive integer. Put x= x+ λy in (9) we get

$\mathcal{Z} \ni \lambda\{[\varphi(x), y] + C_1^n[h_1(x;y), x]\} + \lambda^2\{C_1^n[h_1(x;y),y] + C_2^n[h_2(x;y),x]\} + \ldots + \lambda^n\{[\varphi(y),x] + C_{n-1}^n[h_{n-1}(x;y),y]\}$,∀x, y∈ $\mathcal{R}$ (10)

$\Rightarrow [\varphi(x), y] + n[h_1(x;y), x] \in \mathcal{Z}$ (∵ Lemma 2.4 and (10)) ∀x, y∈ $\mathcal{R}$ **(11)**

Put y = x^2 in (11), we can show that

$[\varphi(x),x^2] + n[h_1(x;x^2), x] = (2n+2)[\varphi(x),x]x$, ∀x∈ $\mathcal{R}$ (12)

Commuting with φ(x) in (12) we get $(2n+2)[\varphi(x), x]^2 = 0$, ∀x∈ $\mathcal{R}$ (13)

Further by substituting y = xy in (11) and on simplifying and reducing we get

$(2n+1)[\varphi(x), x]^2 = 0$, ∀x∈ $\mathcal{R}$ (14)

Now from (13) and (14) we get relation $[\varphi(x), x]^2 = 0$, ∀x∈ $\mathcal{R}$

⇒ $[\varphi(x), x] = 0$, ∀x∈ $\mathcal{R}$ (∵ the center of a semi prime ring contains no nonzero Nilpotent elements)

Therefore φ is commute on R.

Now, the main result, which is an analogue of Posner's theorem [5, Theorem 2], is as follows:

Theorem 2.6: Let n ≥2 be a fixed positive integer and let $\mathcal{R}$be an n!-torsion free prime ring. Suppose that there exists a nonzero symmetric n-derivation $\Phi: \mathcal{R}^n \longrightarrow \mathcal{R}$ such that the trace φ of Φ is centralizing on $\mathcal{R}$. Hence $\mathcal{R}$ is commutative.

Proof: Suppose that $\mathcal{R}$ is non-commutative. From the Theorem 2.5, φ is commuting on$\mathcal{R}$. By Theorem 2.3, Φ = 0 which is a contradiction.This gives the conclusion of the theorem.

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Dr. Dhananjaya Reddy/Lecturer In Mathematics/Govt. Degree College/ Puttur/ Chittoor (Dt/ A.P.