ONE MODULO THREE MEAN LABELING OF SOME SPECIAL GRAPHS

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Abstract: The concept of mean labeling was introduced by Somasundaram and Ponraj. Different kinds of mean labeling are further studied by Gayathri and Gopi . Swaminathan and Sekar introduced the concept of modulo three graceful labeling. As an analogue, Jayanthi and Maheswari introduced one modulo three mean labeling and proved that some standard graphs are one modulo three mean graphs. We have already proved some necessary conditions and properties for one modulo three mean labeling and verified one modulo three meanness for some family of trees. In this paper, we obtain one modulo three mean labeling of some special graphs.

Keywords: Mean graphs, Mean labeling, One modulo three mean graphs (OMTMG), One modulo three mean labeling (OMTML).

Introduction: All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [8]. The symbols $V(G)$ and *E(G)* will denote the vertex set and edge set of a graph G. Labeled graphs serve as useful models for a broad range of applications
[1], [2].
Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research in 1967 [10].
Somasundaram and Ponraj [11] have introduced the concept of mean labeling. Different kinds of mean labeling are studied by Gayathri and Gopi in [4], [5]. Swaminathan and Sekar [12] introduced the concept of modulo three graceful labeling. As an analogue, Jayanthi and Maheswari [9] introduced one modulo three mean labeling and proved that some standard graphs are one modulo three mean graphs. In references [6], [7], we have proved some necessary conditions and properties for one modulo three mean labeling and established that some family of trees. In this paper, we obtain one modulo three mean labeling of some special graphs.

2. Prior Results

Observation 2.1 [6]

(i) $3q-2 \equiv 1(mod\,3)\,\forall q$

(ii) $3q-2$ is odd if *q is odd*

$3q-2$ is even if *q is even*

(iii) $3q-2 \equiv \begin{cases} 1(\mathrm{mod}\,6) & \text{if } q \text{ is odd} \\ 4(\mathrm{mod}\,6) & \text{if } q \text{ is even} \end{cases}$

(iv) $3q-3$ is even if *q* is odd and

$3q-3$ is odd if *q* is even

(v) $3q-3 \equiv \begin{cases} 0(\mathrm{mod}\,6) & \text{if } q \text{ is odd} \\ 3(\mathrm{mod}\,6) & \text{if } q \text{ is even} \end{cases}$

Property 2.2 [9]

Let $G=(p,q)$ be a one modulo three mean graph with one modulo three mean labeling *f*. Let *t* be the number of edges whose one vertex label is even and the other is odd.Then $\sum_{v\in V(G)} d(v)f(v)+t=q(3q-1)$ where $d(v)$ denotes the degree of a vertex *v*.

In [6], we have obtained the results listed below.

Property 2.3

If a graph *G* is a one modulo three mean graph then 0 and 1 are vertex labels.

Property 2.4

If a graph $G=(p,q)$ is a one modulo three mean graph then $3q-3$ and $3q-2$ are ought to be the vertex labels.

Corollary 2.5: If $G=(p,q)$ is a one modulo three mean graph with one modulo three mean labeling *f*, then $\sum_{v\in V(G)} d(v)f(v)\geq q^2$

Property 2.6

Let $G=(p,q)$ be a*l* regular one modulo three mean graph with *l* even. Let *t* be the number of edges whose one vertex label is even and other is odd then *t* is even.

Property 2.7

Let $G=(p,q)$ be a one modulo three mean graph.

(i) If *q* is odd then $0,1,3q-2$ cannot be the vertex labels of the cycle C_3 contained in *G*.

(ii) If *q* is even then $0,1,3q-3$ cannot be the vertex labels of the cycle C_3 contained in *G*.

(iii) If *q* is odd then $1,3q3,3q-2$ cannot be the vertex labels of the cycle C_3 contained in *G*.

(iv) If *q* is even then $0,3q-3,3q-2$ cannot be the vertex labels of the cycle C_3 contained in *G*.

Theorem 2.8

Let *G* be a connected one modulo three mean graph. Let *u* be a vertex with label $f(u)$. Let *v* be any vertex adjacent to the vertex *u* with label $f(v)$

a) If $f(u)\equiv 0\ (mod\,6)$ then

$f(v) \equiv 1 \ (\text{mod } 6)$

b) If $f(u) \equiv 1 \ (mod\, 6)$ then

$f(v) \equiv 0 \ (or) 1 \ (mod\, 6)$

c) If $f(u) \equiv 3 \ (mod\, 6)$ then

$f(v) \equiv 4 \ (\text{mod } 6)$

d) If $f(u) \equiv 4 \ (mod\, 6)$ then

$f(v) \equiv 3 (or) 4 \ (\text{mod } 6)$

Theorem 2.9

If G is a connected one modulo three mean graph then all its vertices receive labels as either $0 \ (or) 1 \ (mod\, 6)$.

Theorem 2.10

If $G=(p,q)$ is a connected one modulo three mean graph then q is odd.

Corollary 2.11

If G is a connected graph with q even then it is not a one modulo three mean graph.

Theorem 2.12

If $G=(p,q)$ is a tree of odd order then it is not a one modulo three mean tree.

Theorem 2.13

If G is a unicyclic connected graph with q even then it is not a one modulo three mean graph.

Theorem 2.14

If G is a connected one modulo three mean graph of odd size and then $\Delta \leq \frac{q+1}{2}$ Where Δ is the maximum degree of a vertex in G.

3. Main Results

Definition 3.1

A graph $G=(p,q)$ is said to be *one modulo three mean graph* (OMTMG) if there is an injective function f from the vertex set of G to the set $\{0,1,3,4,\ldots,3q-5,3q-3,3q-2\}$ with f is one-one and f induces a bijection f^* from the edge set of G to the set $\{1,4,7,10,..,3q-5,3q-2\}$ where $f^*(uv)=\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ and the function f is called *one modulo three mean labeling* (OMTML) of G. Here, $f^*(uv) \equiv 1 (mod\, 3)$ for every edge uv in G.

Definition 3.2

Let G be a graph. Let G' be a copy of G. The ***mirror graph*** $M(G)$ of G is defined as the disjoint union of G and G' with additional edges joining each vertex of G to its corresponding vertex in G'

Theorem 3.3

The mirror graph $M(P_n)$ is a one modulo three mean graph if and only if n is odd.

Proof:

Assume n is odd.

Let $\{v_i, v_i', 1 \leq i \leq n\}$ be the vertices and $\{e_i, 1 \leq i \leq 3n-2\}$ be the edges of $M(P_n)$ which are denoted as in Fig.3.1.

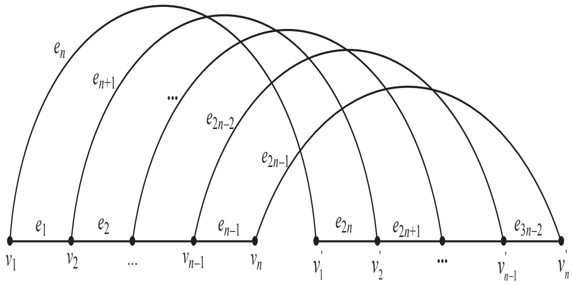

Fig.3.1: Ordinary labeling of $M(P_n)$

We know that. $|V(M(P_n))| = 2n$

and $|E(M(P_n))| = 3n-2$

First we label the vertices as follows:

Define $f: V \to \{0, 1, 3, 4, \ldots, 3q-3, 3q-2\}$

$$f(v_{2i-1}) = 6(i-1), 1 \leq i \leq \frac{n+1}{2}$$

$$f(v_{2i}) = 6(i-1)+1, 1 \leq i \leq \frac{n-1}{2}$$

$$f(v_{2i-1}') = 6(n-1) + 6(i-1) + 1, 1 \leq i \leq \frac{n+1}{2}$$

$$f(v_{2i}') = 6n + 6(i-1), 1 \leq i \leq \frac{n-1}{2}$$

Then the induced edge labels are

$$f^*(e_i) = 3i - 2, 1 \leq i \leq 3n - 2$$

The above defined function f provides one modulo three mean labeling of the graph $M(P_n)$. Hence the graph $M(P_n)$ is a one modulo three mean graph when n is odd.

Conversely, let us assume that $M(P_n)$ is a one modulo three mean graph. Suppose n is even, then the number of edges in $M(P_n)$ is $q=3n-2$ are even. Now, by Corollary 2.11, $M(P_n)$ is not a one modulo three mean graph, a contradiction to our assumption. Hence n is odd.

One modulo three mean labeling of the graph $M(P_5)$ is shown in Fig.3.2.

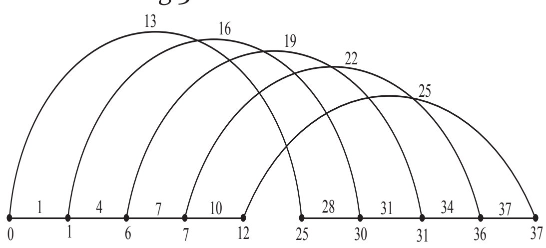

Fig.3.2: OMTML of $M(P_5)$

Definition 3.4
For a graph *G*, the *split graph* is obtained by adding to each vertex *v*, a new vertex *v'* such that *v'* is adjacent to every vertex that is adjacent to *v* in G. The resultant graph is denoted as $spl(P_n)$

Theorem 3.5
The $spl(P_n)$ is a one modulo three mean graph if and only if n is even.

Proof:
Assume n is even
Let $\{u_i, v_i\ , 1 \le i \le n\}$ be the vertices and $\{e_i, 1 \le i \le n-1\ , e_i^{'}, 1 \le i \le 2(n-1)\}$ be the edges of $spl(P_n)$ which are denoted as in Fig.3.3.

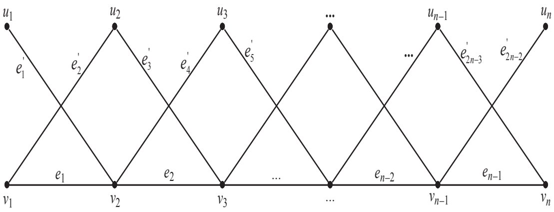

Fig.3.3: Ordinary labeling of $spl(P_n)$

We know that $\left|V\left(spl\left(P_n\right)\right)\right| = 2n$

and $\left|E\left(spl\left(P_n\right)\right)\right| = 3(n-1)$

First we label the vertices as follows:

Define $f : V\left(spl\left(P_n\right)\right) \to \{0,\ 1,\ 3,\ 4,\ ...,3q-3,\ 3q-2\}$

For $1 \le i \le n$,

$$f(v_i) = \begin{cases} 9(i-1) & i \text{ is odd} \\ 9i-11 & i \text{ is even} \end{cases}$$

$$f(u_i) = \begin{cases} 9i-3 & i \text{ is odd} \\ 9i-17 & i \text{ is even} \end{cases}$$

Then the induced edge labels are

$$f^*\left(e_i\right) = 9i - 5\ , 1 \le i \le n-1$$

For $1 \le i \le 2(n-1)$

$$f^*\left(e_i^{'}\right) = \begin{cases} \dfrac{9i-7}{2} & i \text{ is odd} \\ \dfrac{9i-4}{2} & i \text{ is even} \end{cases}$$

The above defined function *f* provides one modulo three mean labeling of the graph $spl(P_n)$. Hence the graph $spl(P_n)$ is a one modulo three mean graph when n is even.

Conversely, let us assume that $spl(P_n)$ is a one modulo three mean graph. Suppose *n* is odd, then the numbers of edges $q = 3(n-1)$ are even. Now, by Corollary 2.11, $spl(P_n)$ is not a one modulo three mean graph, a contradiction to our assumption.

Hence *n* is even.

One modulo three mean labeling of the graph $spl(P_{10})$ is shown in Fig.3.4.

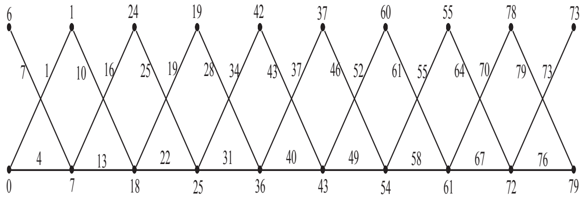

Fig.3.4: OMTML of $spl(P_{10})$

Definition 3.6: Consider two copies of Graph *G* (wheel, star, fan and friendship) namely G_1 and G_2. Then the graph $G' = \langle G_1 \# G_2 \rangle$ is the graph obtained by joining the apex vertices of G_1 and G_2 by an edge as well as to a new vertex *v'*.

Theorem 3.7
The graph $\langle K_{1,n} \# K_{1,n} \rangle$ is a one modulo three mean graph for all *n*.

Proof:
Let $\{u, v, w, u_i, v_i, 1 \le i \le n\}$ be the vertices and $\{a_i, b_i, 1 \le i \le n, c, d, e\}$ be the edges of graph $\langle K_{1,n} \# K_{1,n} \rangle$ which are denoted as in Fig.3.5

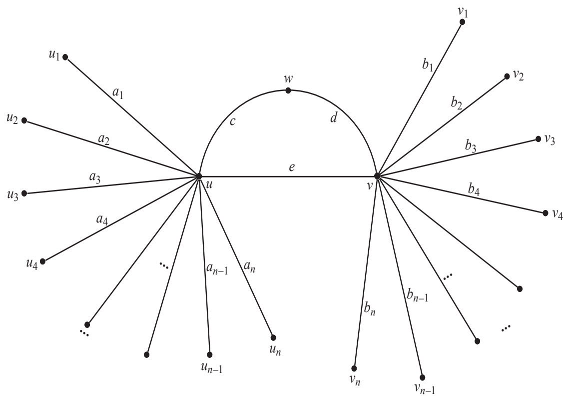

Fig.3.5: Ordinary labeling of $\langle K_{1,n} \# K_{1,n} \rangle$

First we label the vertices as follows:

Define $f : V \to \{0,\ 1, 3, 4, 6, 7, ..., 3q-3,\ 3q-2\}$

$$f(u) = 1\ ; \qquad f(v) = 6(n+1)$$

$$f(w) = 6n + 1;$$

$$f(v_n) = 6(n+1) + 1$$

$$f(u_i) = 6(i-1)\ , 1 \le i \le n$$

$$f(v_i) = 6i + 1, 1 \le i \le n - 1$$

Then the induced edge labels are

$$f^* : E(G) \to \{1,\ 4,\ 7,\ 10,\ ...,\ 3q-2\}$$

$$f^*(c) = 3n + 1 \qquad f^*(e) = 3(n+1) + 1$$

$$f^*(d) = 6(n+1) - 2$$

$f^*(b_n) = 6(n+1)+1$

$f^*(a_i) = 3i-2 \ ,1 \le i \le n$

$f^*(b_i) = 3i+3(n+1)+1 \ ,1 \le i \le n-1$

The above defined function f provides one modulo three mean labeling of $\langle K_{1,n} \# K_{1,n} \rangle$. Hence the graph $\langle K_{1,n} \# K_{1,n} \rangle$ is one modulo three mean graph.

One modulo three mean labeling of the graph $\langle K_{1,9} \# K_{1,9} \rangle$ is shown in Fig.3.6

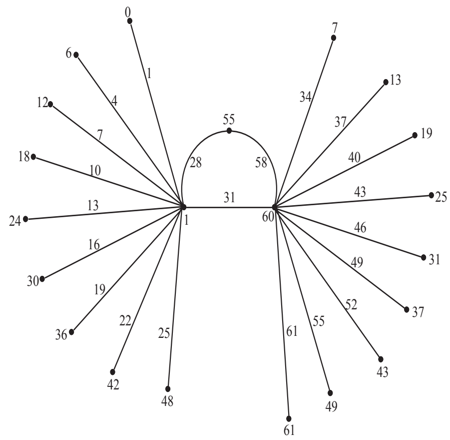

Fig.3.6: Ordinary labeling of $\langle K_{1,9} \# K_{1,9} \rangle$

Theorem 3.8

The graph $\langle K_{1,m} \# K_{1,n} \rangle$ is not a one modulo three mean graph if $m = n+1$.

Proof: Suppose $G = \langle K_{1,m} \# K_{1,n} \rangle$ is a one modulo three mean graph. The number of edges in $\langle K_{1,m} \# K_{1,n} \rangle$ is $q=m+n+3$ are even, a contradiction to Corollary 2.11. Hence the theorem.

Theorem 3.9: The graph $\langle K_{1,m} \# K_{1,n} \rangle, \ m \ge n+2$ is not a one modulo three mean graph.

Proof: Suppose $\langle K_{1,m} \# K_{1,n} \rangle, \ m \ge n+2$ is a one modulo three mean graph.

Case 1: m even, n odd (or) m odd, n even

In this case, the number of edges in $q=m+n+3$ is even a contradiction to Corollary 2.11.

Case 2: m even, n even (or) m odd, n odd

In this case, the number of edges in $q=m+n+3$ is odd. Therefore by Theorem 2.14, $\Delta(G) \le \frac{q+1}{2}$.

Here, G= $\langle K_{1,m} \# K_{1,n} \rangle$, $\Delta = m+2$,

$q = m+n+3$

$$m+2 \le \frac{m+n+3+1}{2}$$

$$2m+4 \le m+n+4$$

$$m \le n$$

a contradiction to $m \ge n+2$.

Theorem 3.10: The graph $\langle K_{1,m} \# K_{1,n} \rangle$ is a one modulo three mean graph if and only if $m=n$.

Proof: Follows from Theorems 3.7, 3.8 and 3.9.

Definition 3.11: The Cartesian Product of two paths P_m and P_n is known as a ***grid graph***. It is denoted by $P_m \times P_n$.

Theorem 3.12: The planar grid $P_m \times P_n$ is a one modulo three mean graph if m even and n odd (or) m odd and n even.

Proof: Let $V(P_m \times P_n) = \{a_{ij} : 1 \le i \le m, \ 1 \le j \le n\}$

$E(P_m \times P_n) = \{a_{ij}a_{ij+1} : 1 \le i \le m, \ 1 \le j \le n\} \cup \{a_{ij}a_{(i+1)j} : 1 \le i \le m, \ 1 \le j \le n\}$

which are denoted as in the Fig.3.7.

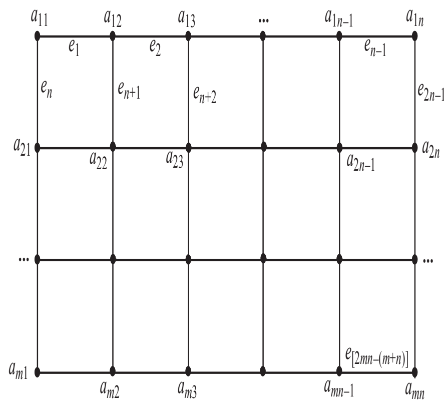

Fig.3.7: Ordinary labeling of $P_m \times P_n$

We know that $|V(P_m \times P_n)| = mn$ and $|E(P_m \times P_n)| = 2mn-(m+n)$.

First we label the vertices as follows:

Define $f : V(P_m \times P_n) \to \{0, 1, 3, 4, ..., 3q-3, 3q-2\}$

For $1 \le i \le m$ *and* $1 \le j \le n$

when i and j both are odd then

$f(v_{ij}) = 3(j-1)+3(i-1)(2n-1)$

when i is odd and j is even then

$f(v_{ij}) = 3(j-2)+3(i-1)(2n-1)+1$

when i is even and j is odd then

$f(v_{ij}) = 3(j-1)+3(i-2)(2n-1)+6(n-1)+1$

when i and j both are even

$f(v_{ij}) = 3(j-2)+3(i-2)(2n-1)+6n$

Then the induced edge labels are

$f^*(e_i) = 3i-2, 1 \le i \le 2mn-(m+n)$

The above defined function f provides one modulo three mean labeling of the graph $P_m \times P_n$. Hence $P_m \times P_n$ is a one modulo three mean graph.

One modulo three mean labeling of the graph $P_6 \times P_5$ is shown in Fig.3.8.

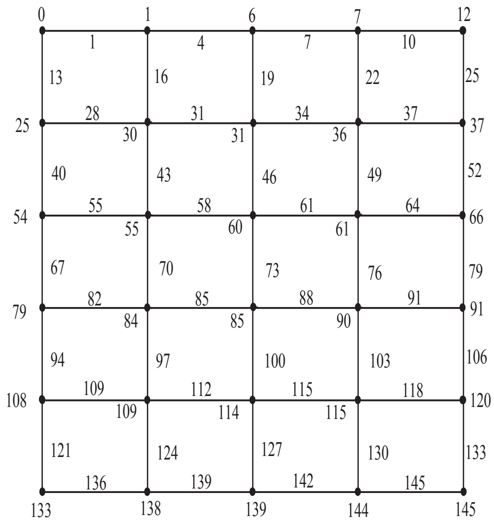

Fig.3.8: OMTML of $P_6 \times P_5$

Theorem 3.13: The planar grid $P_m \times P_n$ is not a one modulo three mean graph if both m and n are odd (or) even.

Proof: Suppose the graph $P_m \times P_n$ is a one modulo three mean graph. The number of edges in $P_m \times P_n$ is $q = 2mn - (m+n)$ are even, a contradiction to Corollary 2.11.

Hence the theorem .

Theorem 3.14: The planar grid $P_m \times P_n$ is a one modulo three mean graph if and only if *m* even and *n* odd (or) *m* odd and *n* even.

Proof: Follows from Theorems 3.12 and 3.13.

Definition 3.15: The graph $L_n = P_n \times P_2$ is called ***ladder***.

Theorem 3.16: The graph $L_m \square \overline{K_n}$ is a one modulo three mean graph if and only if m is odd for all n.

Proof: Assume m is odd.

Let $V\left(L_m \square \overline{K_n}\right) = \{u_i, v_i, 1 \le i \le m\} \cup$

$\{u_{ij}, v_{ij}, 1 \le i \le m, 1 \le j \le n\}$

$E\left(L_m \square \overline{K_n}\right) = \{a_i, 1 \le i \le m\} \cup$

$\{b_i, c_i, 1 \le i \le m-1\} \cup$

$\{e_{ij}, f_{ij}, 1 \le i \le m, 1 \le j \le n\}$ which are denoted as in the Fig. 3.9.

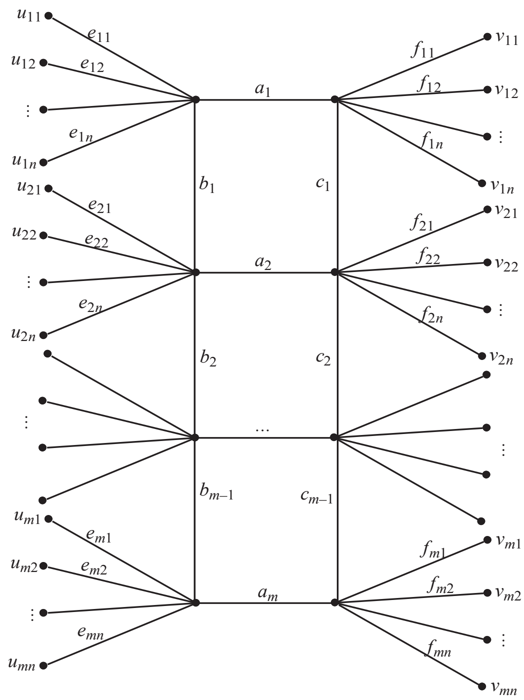

Fig.3.9: Ordinary labeling of $L_m \square \overline{K_n}$

We know that $\left|V\left(L_m \square \overline{K_n}\right)\right| = 2m(n+1)$ and $\left|E\left(L_m \square \overline{K_n}\right)\right| = 2mn + 3m - 2$

First we label the vertices as follows:

Define $f : V \to \{0, 1, 3, 4, \ldots, 3q-3, 3q-2\}$

For $1 \le i \le m$ *and* $1 \le j \le n$

when *i* is odd

$$f(u_{ij}) = (6n+9)(i-1) + 6(j-1)$$

$$f(v_{ij}) = (6n+9)(i-1) + 6j + 1$$

$$f(u_i) = (6n+9)(i-1) + 1$$

$$f(v_i) = (6n+9)(i-1) + 6n$$

when *i* is even

$$f(u_{ij}) = (6n+9)(i-2) + 6(j-1) + 6(n+3) + 1$$

$$f(v_{ij}) = (6n+9)(i-2) + 6(j-1) + 6(n+1)$$

$$f(u_i) = (6n+9)(i-2) + 6(n+1) + 6n$$

$$f(v_i) = (6n+9)(i-2) + 6(n+2) + 1$$

Then the induced edge labels are

For $1 \le i \le m$ *and* $1 \le j \le n$

when *i* is odd

$$f^*(e_{ij}) = (6n+9)(i-1) + 3(j-1) + 1$$

$$f^*(f_{ij}) = (6n+9)(i-1) + 3(j-1) + 3(n+1) + 1$$

when *i* is even

$$f^*(e_{ij}) = (6n+9)(i-2) + 3(j-1) + 9n + 13$$

$f^*\left(f_{ij}\right)=\left(6n+9\right)\left(i-2\right)+3\left(j-1\right)+6\left(n+1\right)+4$ For $1\le i\le m$

$f^*\left(a_i\right)=\left(6n+9\right)\left(i-1\right)+3n+1$

For $1\le i\le m-1$

$f^*\left(b_i\right)=\left(6n+9\right)\left(i-1\right)+6\left(n+1\right)-2$

$f^*\left(c_i\right)=(6n+9)\ (i-1)+6(n+1)+1$

The above defined function *f* provides one modulo three mean labeling of the graph $L_m \square \overline{K_n}$. Hence the graph $L_m \square \overline{K_n}$ is a one modulo three mean graph when m is odd.

Conversely, let us assume that $L_m \square \overline{K_n}$ is a one modulo three mean graph. Suppose m is even, then the number of edges in $L_m \square \overline{K_n}$ is $q=2mn+3m-2$ are even. Now, by Corollary 2.11, $L_m \square \overline{K_n}$ is not a one modulo three mean graph, a contradiction to our assumption.

Hence m is odd.

One modulo three mean labeling of the graph $L_5 \square \overline{K_4}$ is shown in Fig.3.10.

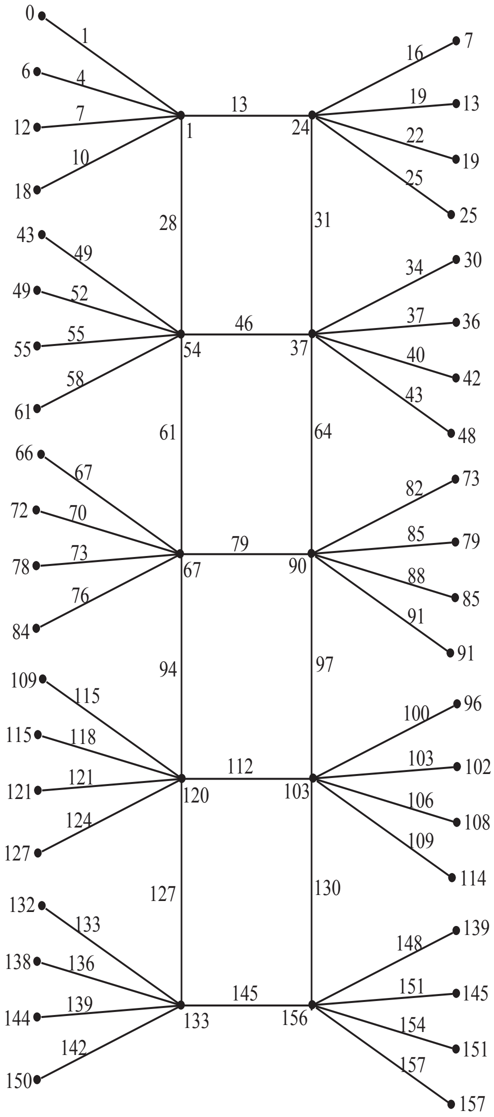

Fig.3.10: OMTML labeling of $L_5 \square \overline{K_4}$

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