DIRECTED EVEN-EDGE-GRACEFUL LABELING OF WHEEL GRAPH

B. GAYATHRI, A. JOEL SURESH

Abstract: A (p,q) graph G is said to be directed even-edge-graceful if there exists an orientation of G and a labeling f of the arcs A of G with $\{1,2,3,...,2q\}$ such that the induced mapping g on V defined by $g(v)=[f^{+}(v)-f^{-}(v)](mod\,2s)$ are distinct and even, where $f^{+}(v)=$ the sum of the labels of all arcs with head v and $f^{-}(v)=$ the sum of the labels of all arcs with tail v, where $s=max\,(p,q)$. A graph G that admits a directed even-edge-graceful labeling is called a directed even-edge-graceful graph. In this paper, we investigate directed even-edge-gracefulness of wheel graph.

Keywords: directed even-edge-graceful graph, directed even-edge-graceful labeling, wheel.

Introduction: All graph in this paper are finite, simple and directed. Terms not defined here are used in the sense of Harary [7]. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p,q) graph. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graph serve as useful models for a broad range of applications [1], [2]. A good account on graph labeling problems can be found in the dynamic survey of Gallian [6].

A graph G is called a *graceful labeling* if f is an injection from the vertices of G to the set $\{0,1,2,\ldots,q\}$ such that, when each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct. A graph $G(V,E)$ is said to be *edge-graceful* if there exists a bijection f from E to $\{1,2,\ldots,|E|\}$ such that the induced mapping f^{+} from V to $\{0,1,\ldots,|V|-1\}$ given by, $f^{+}(x)=(\sum f(xy))mod(|V|)$ taken over all edges xy incident at x is a bijection.

A necessary condition for a graph G with p vertices and q edges to be edge-graceful is $q(q+1)\equiv\frac{p(p+1)}{2}(mod\ p)$. Gayathri and Duraisamy introduced the concept of even-edge-graceful labeling in [8] and further studied in [9]-[11]. Bloom and Hsu[3]-[5] extended the notion of graceful labeling to directed graphs. Gayathri and Vanitha[12] extended the concept of edge-graceful graphs to directed graphs and further studied in[13], [14]. A (p,q) graph G is said to be *directed edge-graceful* if there exists an orientation of G and a labeling f of the arcs A of G with $\{1,2,..,q\}$ such that the induced mapping g on V defined by $g(v)=\left[f^{+}(v)-f^{-}(v)\right](mod\ p)$ is a bijection where, $f^{+}(v)=$ the sum of the labels of all arcs with head v and $f^{-}(v)=$ the sum of the labels of all arcs with tail v. A graph is said to be *directed edge-graceful graph* if it has a directed edge-graceful labeling.

In this paper, we extend the notion of even-edge-graceful labeling to directed graph. Here we investigate directed even-edge-gracefulness of wheel graph.

2.Main results

Definition 2.1:

The ***Wheel*** W_n is defined as $W_n=C_n+K_1$ where C_n is a cycle of length n and the number of vertices in a wheel is $n+1$.

Theorem 2.2:

The wheel W_n, $n\geq 4$ is directed even-edge-graceful for $q\equiv 0(mod\,8)$

Proof:

Let $V(W_n)=\{v,v_1,v_2,v_3\ldots,v_n\}$ be the set of vertices.

Now, we orient the edges of W_n, such that the arc set A is given by

$$A(W_n)=\{e_i=(v,v_i),1\leq i\leq\frac{q}{2}\}$$
$$\cup\ \{e_{\frac{2i+q-2}{2}}=(v_i,v_{i-1}),2\leq i\leq n\}$$
$$\cup\ \{e_q=(v_1,v_n)\}$$

The orientation of the edges are given as in the Fig. 1

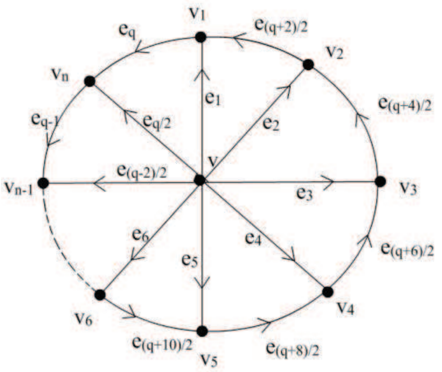


Fig. 1: W_n with orientation

We define edge labels as follows :

$$f(e_i)=2i\ ,1\le i\le \frac{q}{2}\ ;\ f(e_q)=2q\ ;$$

$$f(e_i)=3q-2i,\ \frac{q+2}{2}\le i\le q-1;$$

Then the values of $f^+(v_i), f^-(v_i)$ are computed as under:

$f^+(v_i)=2q\ ,1\le i\le n-1;$

$f^+(v_n)=3q;\quad f^+(v)=0$

$f^-(v_i)=2q+2-2i, 2\le i\le n\ ;$

$f^-(v)=n(n+1)$

$f^-(v_1)=2q;$

The induced vertex labels are

$g(v_i)=2i-2, 2\le i\le n-1$.

$g(v_1)=0, g(v_n)=4n-2, g(v)=3n$

Clearly,

$g(V)=\{0,2,4,..,2n-4,4n-2,3n\}$

$\subseteq\{0,2,4,\ldots,2s-2\}$,

where $s=q=2n$

So, it follows that all the vertex labels are distinct and even. Hence, the wheel W_n is directed even-edge-graceful for all $q\equiv 0(mod\,8)$

The directed even-edge - graceful labeling of W_8 is given in Fig. 2

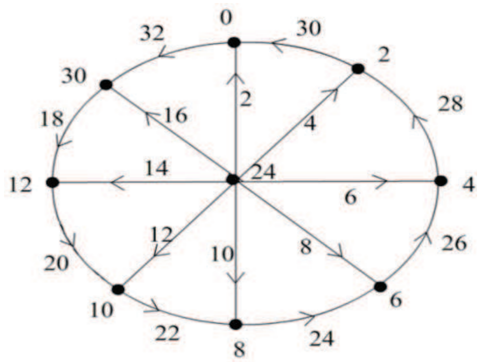


Fig.2 :Directed even-edge-graceful labeling of W_8

Theorem 2.3:

The wheel graph W_n is directed even-edge-graceful for $q\equiv 2(mod\,8)$

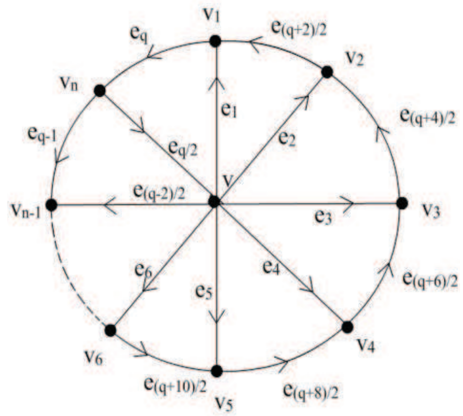
Proof:

Let $V(W_n)=\{v,v_1,v_2,v_3\ldots,v_n\}$ be the set of vertices. Now, we orient the edges of W_n, such that the arc set A is given by

$$A(W_n)=\{e_i=(v,v_i), 1\le i\le \frac{q}{2}-1\}$$

$$\cup\ \{e_{\frac{2i+q-2}{2}}=(v_i,v_{i-1}), 2\le i\le n\}$$

$$\cup\ \{e_q=(v_1,v_n)\}\cup\{e_{\frac{q}{2}}=(v_n,v)\}$$



The edges and their orientation are given in Fig. 3

Fig. 3: W_n with orientation

We define edge labels as follows :

$$f(e_i)=2i+2,\ 1\le i\le \frac{q-2}{2};$$

$$f(e_i)=3q-2i-2,\ \frac{q+2}{2}\le i\le q-2\ ;$$

$$f(e_q)=2q-2;$$

$$f(e_{\frac{q}{2}})=2\ ;\qquad f(e_{q-1})=2q\ ;$$

Then the values of $f^+(v_i), f^-(v_i)$ are computed as under.

$f^+(v_i)=2q\ ,1\le i\le n-2;$

$f^+(v_{n-1})=3q;\quad f^+(v_n)=2q-2$

$f^+(v)=2;\quad f^-(v_1)=2q-2$

$f^-(v_{n-1})=2n+2$

$f^-(v_n)=2q+2;$

$f^-(v_i)=2q-2i, 1\le i\le n-2$

$f^-(v)=n(n+1)-2$

The induced vertex labels are

$g(v_i)=2i,\ 1\le i\le n-2$,

$g(v_{n-1})=2q-2$;

$g(v_n)=2q-4,\quad g(v)=q+4$

Clearly,

$g(V)=\{2,4,..,2n-4,2q-2,2q-4,q+4\}$

$\subseteq\{0,2,4,\ldots,2s-2\}$, *where* $s=q=2n$

So, it follows that all the vertex labels are distinct and even. Hence, the wheel W_n is directed even-edge-graceful for all $q\equiv 2(mod\,8)$

The directed even-edge - graceful labeling of W_5 is given in Fig. 4.

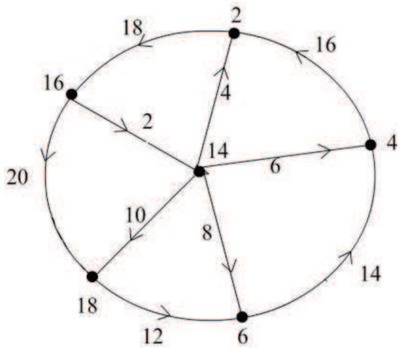


Fig. 4:Directed even-edge-graceful labeling of W_5

Theorem 2.4:

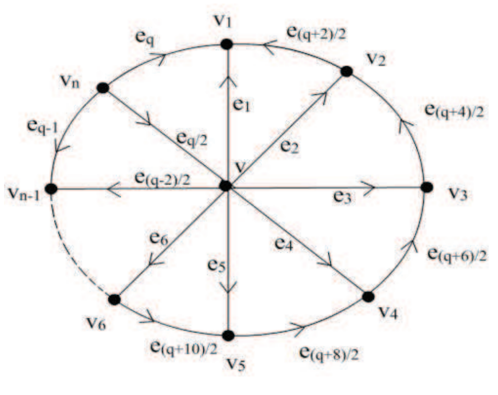
The wheel graph W_n is directed even-edge-graceful for $q\equiv 6(mod\,8)$.

Proof: Let $V(W_n)=\{v,v_1,v_2,v_3\ldots,v_n\}$ be the set of vertices. Now, we orient the edges of W_n, such that the arc set A is given by

$A(W_n)=\{e_i=(v,v_i),1\le i\le \frac{q}{2}-1\}$

$\cup\ \{e_{\frac{2i+q-2}{2}}=(v_i,v_{i-1}),2\le i\le n\}$

$\cup\ \{e_q=(v_n,v_1)\}\cup\ \{e_{\frac{q}{2}}=(v_n,v)\}$



The edges and their orientation are given in Fig.5

Fig. 5 : The ordinary labeling of W_n

We define edge labels as follows :

$f(e_1)=2$; $f(e_2)=4$; $f(e_3)=q$;

$f(e_i)=2i-2,\ 4\le i\le\frac{q}{2}$;

$f(e_i)=3q-2i+2,\ \frac{q+8}{2}\le i\le q$;

$f(e_{\frac{q+2}{2}})=2q-2$; $f(e_{\frac{q+4}{2}})=2q-4$; $f(\frac{q+6}{2})=2q$

Then the values of $f^+(v_i),f^-(v_i)$ are computed as under.

$f^+(v_1)=3q+2$; $f^-(v_n)=3q+4$

$f^+(v_2)=2q; f^+(v_3)=2q; f^+(v_4)=3q;$

$f^+(v_2)=2q; f^+(v_3)=2q; f^+(v_4)=3q;$

$f^+(v_i)=2q\ ,5\le i\le n-1;$

$f^+(v_n)=0;\ f^+(v)=q-2;$

$f^-(v_1)=0; f^-(v_2)=2q-2;$

$f^-(v_3)=2q-4; f^-(v_4)=2q;$

$f^-(v_i)=2q-2i+4\ ,5\le i\le n-1;$

$f^-(v)=n(n-1)+2$

The induced vertex labels are

$g(v_1)=q+2; g(v_2)=2;$

$g(v_3)=q+4; g(v_4)=0;$

$g(v_i)=2i-4,5\le i\le n\ ; g(v)=2q-4$

Clearly,

$g(V)=\{0,2,4,..,2n-4,2q-4,q+2\}$

$\subseteq\{0,2,4,\ldots,2s-2\}$, where $s=q=2n$

So, it follows that all the vertex labels are distinct and even. Hence, the wheel W_n is directed even-edge-graceful for all $q\equiv 6(mod\,8)$

The directed even-edge - graceful labeling of W_7 is given in Fig. 6.

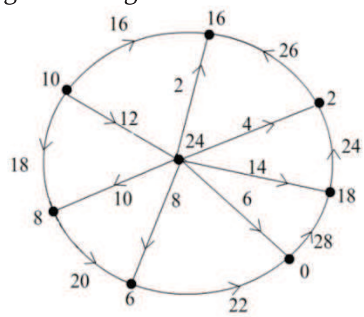


Fig. 6:Directed even-edge-graceful labeling of W_7

Theorem 2.5: The wheel graph W_n is directed even-edge-graceful for $q\equiv 4(mod\,8)$

Proof: Let $V(W_n)=\{v,v_1,v_2,v_3\ldots,v_n\}$ be the set of vertices. Now, we orient the edges of W_n, such that the arc set A is given by

$A(W_n) = \{e_i = (v, v_i), 1 \le i \le \frac{q}{2}\}$

$\bigcup \{e_{\frac{2i+q-2}{2}} = (v_i, v_{i-1}), 2 \le i \le n\}$

$\bigcup \{e_q = (v_1, v_n)\}$

The edges and their orientation are given in Fig. 1
We define edge labels as follows :

$f(e_i) = 2i, 1 \le i \le \frac{q}{4} - 1;$

$f(e_i) = 2(i+1), \frac{q}{4} \le i \le \frac{q}{2} - 2;$

$f(e_{\frac{q}{2}-1}) = 2q - \frac{3q}{2};$

$f(e_i) = 3q - 2i, \frac{q+2}{2} \le i \le \frac{3q-4}{4};$

$f(e_i) = 3q - 2i - 2, \frac{3q}{4} \le i \le q - 2;$

$f(e_{\frac{q}{2}}) = q;\ f(e_{q-1}) = 2q - \frac{q}{2};$

$f(e_q) = 2q;$

Then the values of $f^+(v_i), f^-(v_i)$ are computed as under.

$f^+(v_i) = 2q, 1 \le i \le n-1;$

$f^+(v_n) = 3q; f^+(v) = 0;$

$f^-(v_i) = 2q - 2i + 2, 1 \le i \le \frac{n}{2};$

$f^-(v_i) = 2q - 2i, \frac{n+2}{2} \le i \le n-1$

$f^-(v_n) = 3n;\ f^-(v) = n(n+1)$

The induced vertex labels are

$g(v_n) = 3n, \quad g(v) = n;$

$g(v_1) = 0, \quad g(v_2) = 2, \quad g(v_3) = 4,$

$g(v_i) = 2i - 2, 4 \le i \le \frac{n}{2}$ *and* $(n > 6);$

$g(v_i) = 2(i+1) - 2, \frac{n}{2} + 1 \le i \le n-1$

Clearly,

$g(V) = \{0, 2, 4, .., n-2, n+2, \ldots, 2n-2, n, 3n\}$

$\subseteq \{0, 2, 4, \ldots, 2s-2\}$, *where* $s = q = 2n$

So, it follows that all the vertex labels are distinct and even. Hence, the wheel W_n is directed even-edge-graceful for all $q \equiv 4(mod\, 8)$. The directed even-edge - graceful labeling of W_6 is given in Fig. 6.

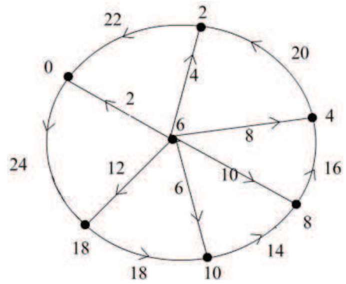


Fig. 6:Directed even-edge-graceful labeling of W_6

Theorem 2.6: The wheel graph W_n is directed even-edge-graceful for $n = 3$

Proof: The directed even-edge-graceful of W_3 is given in Fig. 7

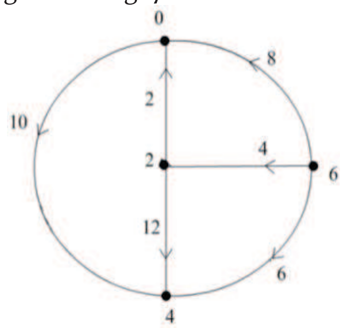


Fig. 7:Directed even-edge-graceful labeling of W_3

Clearly, $g(V) = \{0, 2, 4, 6\} \subseteq \{0, 2, 4, \ldots, 2s-2\}$, where $s = q = 2n$. So, it follows that all the vertex labels are distinct and even. Hence, the wheel W_n is directed even-edge-graceful for $n = 3$.

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B. Gayathri/Associate professor/PG and Research/ Department of Mathematics/
Periyar E.V.R. College (Autonomous)/Tiruchirapplli /620 023.
Joel Suresh/Full Time Research Scholar/PG and Research/Department of Mathematics/
Periyar E.V.R. College (Autonomous)/Tiruchirapplli/ 620 023.