PARTITION DIMENSION OF PYRAMID NETWORK

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Abstract: For a vertex v of a connected graph G and a subset $S$of $V(G)$, the distance between v and $S$is$d(v,S) = min\{d(v,x) \mid x \in S\}$. Let $\prod = \{S_1, S_2 \dots S_k\}$ be an ordered k-partition of $V(G)$. The representation of v with respect to $\prod$ is the k-vector $r(v \mid \prod) = (d(v,S_1), d(v,S_2) \dots d(v,S_k))$. If the k-vectors $r(v \mid \prod)$, for all $v \in V(G)$ are distinct then the k-partition is a resolving partition. The minimum k for which there is a resolving k-partition of $V(G)$ is called the partition dimension $pd(G)$ of G.In this paper, we determine the partition dimension of Pyramid network.

Keywords: Resolving partition, minimum resolving partition, partition dimension, pyramid network.

Introduction: The n-$dimensional pyramid network$, denoted by $PN(n)$, is a hierarchy structure based on mesh networks, and is a generalization of the multigrid network. The vertex set of an n-dimensional pyramid network$PN(n)$ is $V(PN(n)) = \{(x,y,i): 1 \le x,y \le 2^i, 0 \le i \le n\}$. For each fixed $i (0 \le i \le n), V_i = \{(x,y,i): 1 \le x,y \le 2^i\}$ is a set of vertices on level i. The subgraph induced by V_i is a mesh network $G(2^i, 2^i)$. Each $(x,y,i) \in V_i$ is adjacent to four vertices of V_{i+1}, namely$(2x-1, 2y, i+1)$, $(2x, 2y, i+1)$, $(2x-1, 2y-1, i+1)$, $(2x, 2y-1, i+1)$.

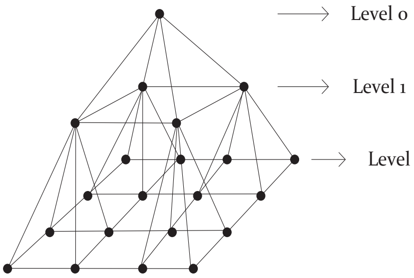

Fig. 1 : Pyramid Network $PN(2)$

The vertex $v_0 = (1,1,0)$ is called the $root$ of $PN(n)$. The number of vertices, edges and diameter of $PN(n)$are $\frac{1}{3}(4^{n+1}-1)$, $4(4^n - 2^n)$and $2n$respectively.

An Overview Of The Paper: One of the fundamental process in graph theory is partitioning the vertex set of a graph into numerous subset based on some prescribed rule. One such rule is the concept of $minimum distance$. The vertices of a connected graph are represented by partitions of vertex set and distances between each vertex and the subsets in the partition are calculated. Based on this concept, resolving partition and partition dimension for a graph was introduced [3]. This concept is applied in the field of chemistry, problems of pattern recognition, image processing and navigation of robots in networks [1], [5], [11].

For $v \in V(G)$ and $S \subset V(G)$, the distance $d(v,S)$ between$v$and S is defined as $d(v,S) = min\{d(v,x) \mid x \in S\}$.The representation of v with respect to $\prod$ is defined as the k-vector$r(v \mid \prod) = (d(v,S_1), d(v,S_2) \dots d(v,S_k))$, where $\prod$ is an ordered k-partition$\{S_1, S_2 \dots S_k\}$ of $V(G)$. The partition $\prod$ is called a $resolving partition$ for G if the distinct vertices of G have distinct representations with respect to $\prod$. The minimum k for which there is a resolving k-partition of $V(G)$ is a $partition dimension pd(G)$of G. A resolving partition of $V(G)$ containing $pd(G)$ elements is called a $minimum resolving partition$ [3], [9].

If G is a nontrivial connected graph, then $pd(G) \le dim(G) + 1$, where $dim(G)$ is a minimum metric dimension of a graph G [4], [5]. For a graph $G$with order $n \ge 2$, $pd(G)$ is 2 if and only if $G = P_n$. The partition dimension $pd(G)$ of a graph G with order n is n if and only if$G = K_n$. Chartrand et al [3] proved that $3 \le pd(G) \le n-1$for a graph Gwhich is neither a path nor a complete graph with order $n \ge 4$[3], [9]. The partition dimension of an n-cycle and Petersen graph are 3 and 4 respectively [3], [9]. Partition dimension problem has been studied for circulant networks [2], hexagonal and honeycomb networks [1], tree [10], cartesian product [7], wheel related graphs [6].

Partition Dimension of Pyramid Network: Level 1 of $PN(1)$ is a cycle on 4 vertices. We know that $pd(C_n) = 3$. To obtain the resolving partition for $PN(1)$, two adjacent vertices in level 1 must be in two different subsets S_i and S_j, $i \ne j$and the other three vertices of $PN(1)$ are in S_k. This observation leads to the following lemma.

Lemma 3. 1 .$pd(PN(1)) = 3$.

Lemma 3.2. Let $G = PN(n), n \ge 2$. Then $pd(G) > 3$.

Proof: Khuller et al [11] has proved that the metric dimension for a n-dimensional mesh is n. Using the fact that $pd(G) \le dim(G) + 1$, we have partition dimension of a mesh is 3.From the definition of Pyramid Network, the subgraph induced by V_i is a mesh network which implies that $pd(G) \ge 3$. Assume that $(x,y,i) \in S_m$ for some m. Only two adjacent vertices in V_{i+1} can belong to S_m and other two adjacent vertices must belong to two different subsets. Since$|V_i| = 4^i, i \ge 1, pd(G) > 3$.

We divide the vertices of level 2 in $PN(2)$ into 4 blocks A, B, C and D. Vertices of each block is adjacent to a unique vertex $v_i, 1 \leq i \leq 4$ in level 1. Every vertex in each block is adjacent to 2 vertices in the same block. We label the vertices of each block A, B, C and D as a_i, b_i,c_i and d_i, $1 \leq i \leq 4$ respectively. See Fig 2.

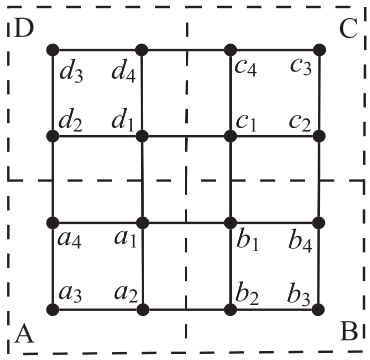

Fig 2 : Partition of layer 2 and its labeling in $PN(2)$

Lemma 3. 3. Let $G = PN(2)$. Then $pd(G) = 4$.

Proof: It can be seen that$d(a_i, c_i) = d(b_i, d_i)$, $d(a_i, B) = d(c_i, B)$, $d(a_i, D) = d(c_i, D)$. From these calculations we infer that the vertices in blocks A and C must belong to different sets and vertices in blocks B and D must belong to different sets. Thus, the vertices of level 2 must be partitioned into atleast 3 sets. Let$V(A) \in S_1$, $V(B) \in S_2$, $V(C), V(D) \in S_3$. For$1 \leq j \leq 3$, $d(v_1, S_j) = (1,2,2)$, $d(v_2, S_j) = (2,1,2)$, $d(v_3, S_j) = (3,2,1)$, $d(v_4, S_j) = (2,3,1)$, where v_i, $1 \leq i \leq 4$ are the vertices of level 1 and v_0 is a root vertex. For any $v_i \in S_j$, the representation of v_i will be same as the representation of some vertex in S_j. Thus $v_i, v_0 \in S_4$. Hence $\prod = \{S_1, S_2, S_3, S_4\}$, where $r(v|\prod)$, for all $v \in V(G)$, are distinct.

There are 4 blocks A, B, C and D in level 3 of $PN(3)$. Each block is a 4×4mesh consisting of 16 vertices. We label them as a_i, b_i, c_i and d_i, $1 \leq i \leq 16$. We observe that $d(a_i, c_i) = d(b_i, d_i)$, $1 \leq i \leq 16$, and$d(a_i, B) = d(c_i, B)$, $d(a_i, D) = d(c_i, D)$,$d(b_i, A) = d(d_i, A)$, $d(b_i, C) = d(d_i, C)$.

Lemma 3. 4. Let $G = PN(3)$. Then $pd(G) = 4$.

Proof: Using the same argument as in lemma 3. 3, the vertices of adjacent blocks must belong to different set in level 3. Each block in level 3 of $PN(3)$ is a 4×4 mesh. To get the distinct representation for each vertex in the level 3, we require at least 3 subsets. Let $V(A) \in S_1, V(B) \in S_2, V(C), V(D) \in S_3$.

According to the above partition of the level 3, the pairs of vertices $\{d_5, d_{14}\}$, $\{d_6, d_{15}\}$, $\{d_7, d_8\}$, $\{d_9, d_{10}\}$ in $D$are equidistant from S_1 and S_2. Similarly, the pairs of vertices $\{c_5, c_{14}\}$, $\{c_6, c_{15}\}$, $\{c_7, c_8\}$, $\{c_9, c_{10}\}$ in $C$are equidistant from S_1 and S_2. If at least one vertex from each pair belongs to different set other than S_1, S_2, S_3, then the representation of each vertex of level 3 will be distinct. Thus$d_5, d_6, d_7, d_9, c_5, c_6, c_7, c_9$ will be in S_4.

It can be easily seen that the maximum distance between a vertex v_i, $1 \leq i \leq 16$ in level 2 and any block in level 3 is 5. In case $v_i \in S_4$, then the representation of v_i with respect to $\{S_1, S_2, S_3, S_4\}$ is not identical with the representations of d_i and c_i, where $d_i, c_i \in S_4$.

Let the vertices of level 0, level 1, level 2 and $\{d_5, d_6, d_7, d_9, c_5, c_6, c_7, c_9\} \in S_4$. Thus, $\prod = \{S_1, S_2, S_3, S_4\}$ is a resolving partition of $pd(PN(3))$ and which is a minimum resolving 4-partition of $pd(PN(3))$. Hence, $pd(PN(3)) = 4$.

In $PN(n)$, level n consists of $2^n \times 2^n$ vertices. There are four blocks A, B, C and D in the level $n$where each block is a$2^{n-1} \times 2^{n-1}$ mesh.

Each $2^{n-1} \times 2^{n-1}$ block can be divided into 2^{2n-8} number of $2^3 \times 2^3$ meshes. The total number of $2^3 \times 2^3$ meshes in each row and column is 2^{n-4}. See Fig 3.

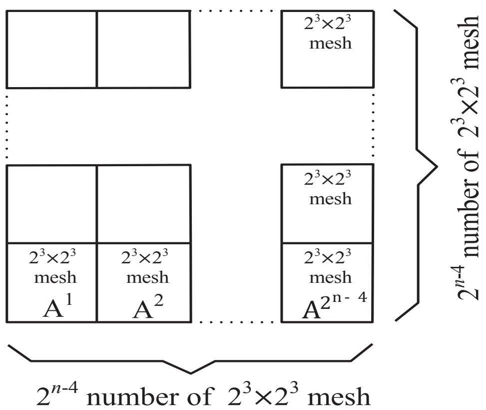

Fig 3 :2^{2n-8} number of $2^3 \times 2^3$ mesh of block A in level n

Each $2^3 \times 2^3$ meshes are labeled row wise as $A^i, 1 \leq i \leq 2^{2n-8}$. The number of $2^2 \times 2^2$ meshes in a block of $2^3 \times 2^3$ is 4 which are labeled as $A^i_j, 1 \leq i \leq 2^{2n-8}, 1 \leq j \leq 4$ and the vertices are labeled as $a^i_{j,k}, 1 \leq k \leq 16$.

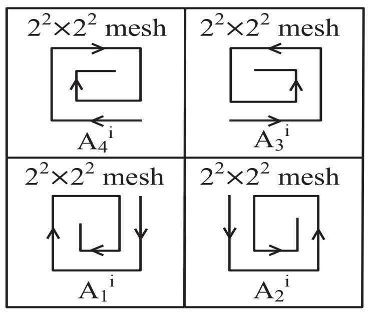

Fig. 4: Four blocks of $2^2 \times 2^2$mesh in A^i with directions to label all four blocks

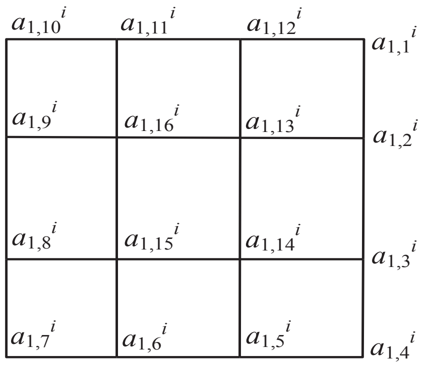

Fig. 5: Labeling of A_1^i.

Similarly, the labels for blocks B, C and $D$are B_j^i, C_j^i and D_j^i respectively and its vertices are $b_{j,k}^i$, $c_{j,k}^i$ and $d_{j,k}^i$, where $1 \leq i \leq 2^{2n-8}, 1 \leq j \leq 4, 1 \leq k \leq 16$. See Fig. 4 and Fig. 5.

Theorem 3. 5. Let $G = PN(n)$. Then $pd(G) > 2^{n-3} + 2$.

Proof: Consider any particular A_j^i. As $d(a_{1,k}^i, A_2^i) = d(a_{3,k}^i, A_2^i)$, $d(a_{1,k}^i, A_4^i) = d(a_{3,k}^i, A_4^i)$, adjacent blocks of A_j^i, $1 \leq j \leq 4$, must not be in the same set. We partition the vertices of A_j^ias $A_1^i \in S_1$, $A_2^i \in S_2$, $A_3^i, A_4^i \in S_3$. While partitioning this way, the pairs of vertices $\{(a_{3,5}^i, a_{3,14}^i), (a_{3,6}^i, a_{3,15}^i), (a_{3,7}^i, a_{3,8}^i), (a_{3,9}^i, a_{3,10}^i)\}$ in A_3^i and $\{(a_{4,5}^i, a_{4,14}^i), (a_{4,6}^i, a_{4,15}^i), (a_{4,7}^i, a_{4,8}^i), (a_{4,9}^i, a_{4,10}^i)\}$ in A_4^i are of equi-distant from $A_1^i$and A_2^i.

For $j = 1, 2$, we observe that $d(a_{3,5}^i, A_j^i) = d(a_{3,14}^i, A_j^i)$, $d(a_{3,6}^i, A_j^i) = d(a_{3,15}^i, A_j^i)$, $d(a_{3,7}^i, A_j^i) = d(a_{3,8}^i, A_j^i)$, $d(a_{3,9}^i, A_j^i) = d(a_{3,10}^i, A_j^i)$, $d(a_{4,5}^i, A_j^i) = d(a_{4,14}^i, A_j^i)$, $d(a_{4,6}^i, A_j^i) = d(a_{4,15}^i, A_j^i)$, $d(a_{4,7}^i, A_j^i) = d(a_{4,8}^i, A_j^i)$, $d(a_{4,9}^i, A_j^i) = d(a_{4,10}^i, A_j^i)$. Thus, one of the vertex from each pair must belong to different set other than S_1, S_2 and S_3.

Consider any two blocks of $2^3 \times 2^3$ mesh, ie, $A^i$and A^{i+1}. The block A^i consists of 4 numbers of $2^2 \times 2^2$blocks A_j^i, $1 \leq j \leq 4$ and the block A^{i+1} also consists of 4 numbers of $2^2 \times 2^2$blocks A_j^{i+1}, $1 \leq j \leq 4$. Consider the block A^i. It can be noted that $d(a_{1,k}^i, A_2^i) = d(a_{3,k}^i, A_2^i)$, $d(a_{1,k}^i, A_4^i) = d(a_{3,k}^i, A_4^i)$ and $max\{d(a_{1,k}^i, A_2^i)\} = max\{d(a_{3,k}^i, A_2^i)\} = max\{d(a_{1,k}^i, A_4^i)\} = max\{d(a_{3,k}^i, A_4^i)\} = 4$. Thus, if a particular block A_j^i belongs to some set S_k then its adjacent blocks will belong to different set. The same condition holds for A^{i+1}.

Thus, we partition the vertices of the two blocks A^i, A^{i+1} as follows: $A_1^i \in S_1$, $A_2^i \in S_2$

$A_1^{i+1} \in S_3$, $A_2^{i+1} \in S_4$, $\{A_3^i, A_4^i, A_3^{i+1}, A_4^{i+1}\} \in S_5$. The four pairs of vertices$\{(a_{4,5}^i, a_{4,14}^i), (a_{4,6}^i, a_{4,15}^i), (a_{4,7}^i, a_{4,8}^i), (a_{4,9}^i, a_{4,10}^i)\}$ in $A_4^i$and four pairs of vertices$\{(a_{3,5}^{i+1}, a_{3,14}^{i+1}), (a_{3,6}^{i+1}, a_{3,15}^{i+1}), (a_{3,7}^{i+1}, a_{3,8}^{i+1}), (a_{3,9}^{i+1}, a_{3,10}^{i+1})\}$ in A_3^{i+1} are of equi-distant from the set S_i, $1 \leq j \leq 4$. Also, two pairs of vertices$\{(a_{3,7}^i, a_{3,8}^i), (a_{3,9}^i, a_{3,10}^i)\}$in A_3^i and two pairs of vertices$\{(a_{4,7}^{i+1}, a_{4,8}^{i+1}), (a_{4,9}^{i+1}, a_{4,10}^{i+1})\}inA_4^{i+1}$ are equi-distant from the set S_i, $1 \leq i \leq 4$ and each pair of these vertices are adjacent vertices. Thus if at least one of the vertex from each pair of vertices belong to different set other than S_i, $1 \leq i \leq 5$, then the representation of the vertices of A^i and A^{i+1} are distinct with respect to S_i, $1 \leq i \leq 5$.

When we consider one row of $2^3 \times 2^3$ blocks in the level n of $PN(n)$, it consists of $2(2^{n-4})$ blocks of $2^3 \times 2^3$ meshes, ie, A^i,B^i, $1 \leq i \leq 2^{n-4}$. Each block is composed of 4 numbers of $2^2 \times 2^2$block. The vertices of each block of A^i is partitioned into at least 3 subsets. Two consecutive blocks A^i,A^{i+1}requires at least 5 subsets and2^{n-4} numbers of $2^3 \times 2^3$ blocks requires at least $2(2^{n-4}) + 1$subsets. To make the representation of that particular row distinct, we requireat least$2^{n-3} + 2$ subsets, ie $\prod = \{S_1, S_2 \ldots S_{2^{n-3}+2}\}$. Hence $pd(G) > 2^{n-3} + 2$.

Conclusion: Pyramid network is one of the important structure in parallel computing and image processing. In this paper, we have obtained the bound for partition dimension of Pyramid network.

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