NEW SUB-CLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY SUBORDINATION

KARTHIYAYINI.O, SIVASANKARI.V

Abstract: In this paper, we introduce a new subclass of analytic bi-univalent functions defined by subordinations. We obtain estimates on the general and initial Taylor-Maclaurin coefficient of this class using Faber polynomial approach. Connections to earlier known results are briefly indicated.

Keywords: Bi-univalent functions, Coefficient estimates, Convolution, Faber polynomial expansions, Subordination ,Univalent functions.

Introduction: Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disc

U = $\{z : z \in C and \mid z \mid < 1\}$

and satisfy the normalization condition $f(0) = f'(0) = 1$.

Let $f(z)$ and $g(z)$ be analytic functions in U, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$ if there exist a Schwarz function $w(z)$ in U, such that $f(z) = g(w(z))$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$[13]

In particular, if the function $g(z)$ univalent in U, then the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

For functions $f(z)$ defined by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

we define convolution of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in U.$$

Let S be the class of A consisting of the functions of the form (1.1) which are also univalent in U. According to Koebe one quarter theorem[13], it is well known fact that every function $f \in$ S has an inverse f^{-1}, which is defined by $f^{-1}(f(z)) = z (z \in U)$ and $f(f^{-1}(w)) = w$ for $|w| < 1/4$.

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1). The bi-univalency condition imposed on the functions $f(z) \in A$ makes the behaviour of their coefficients unpredictable. Lewin [11] first investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ .However Netanyahu[4] showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. Ali et al[14] remarked that finding the bounds for $|a_n|$ when $n \geq 4$ is an open problem. Recently, several researchers such as([1,2,8,9,15,20]) introduced and investigated subclasses of bi-univalent functions and obtained the coefficient bounds for $|a_2|$ and $|a_3|$. S.G.Hamidi and J.M.Jahangiri[19] used Faber polynomial coefficient for finding the estimates on the coefficient bounds for the classes of bi-univalent functions. These bounds prove to be better than those estimates provided by Srivastava et al [8] and Frasin and Aouf[2].Hence, motivated by his work , we also used Faber polynomial approach to find the coefficient estimate of our new subclass of bi-univalent functions.

The object of the present paper is to introduce new subclass of bi-univalent functions defined by subordination using convolution and obtain the general and initial coefficient for the same.

Definition.1.1: Letϕ: U→C, be a convex univalent function such that $\phi(0) = 1$ and $\phi(\bar{z}) = \overline{\phi(z)}$, $(z \in U; \mathrm{Re}(\phi(z)) > 0)$.

A function f(z) is said to be in the class $B_{\Sigma}^{\psi,\beta}(\phi)$, if the following conditions are satisfied:

$f(z) \in \sum$ and

$$e^{i\beta} \frac{(f * \psi)(z)}{z} \prec \phi(z)\cos\beta + i\sin\beta, (z \in U),$$

and

$$e^{i\beta} \frac{(f * \psi)(w)}{w} \prec \phi(w)\cos\beta + i\sin\beta, (w \in U);$$

where $\beta \in (-\pi/2, \pi/2)$, and $g = f^{-1}$.

Remark 1.1: If we set $\psi(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1)-\lambda} z^n$, we obtain the class studied by S.P.Goyal and Pranay Goswami[20].

Remark 1.2: If we set $\psi(z)=z+\sum_{n=2}^{\infty}nz^n$ and $\phi(z)=\frac{1+Az}{1+Bz}, -1\le B<A\le 1,$ then this class reduces to $B_{\Sigma}^{0}(A,B)$[20] and further substituting A= 1-2α , $0\le\alpha<1, B=-1 and \beta=0$,we get known class $B_\Sigma(\beta)$ studied earlier by Srivastva et al[8].

Coefficient Bounds For The Class:

Using the Faber Polynomial expansion of functions $f(z)\in A$ of the form (1.1), the coefficients of its inverse map $g=f^{-1}$ may be expressed as ,[5],

$$g(w)=f^{-1}(w)= w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots)w^n$$

where

$$K_{n-1}^{-n}=\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}$$
$$+\frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3$$
$$+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4$$
$$+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5+(-n+2)a_3^2\right]$$
$$+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)a_3a_4\right]$$
$$+\sum_{i\ge 7}a_2^{n-i}V_i,$$

such that V_i with $7\le i\le n$ is a homogenous polynomial in the variables $a_2,a_3,\ldots,a_n$[6].

In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2}K_1^{-2}=-a_2$$
$$\frac{1}{3}K_2^{-3}=2a_2^2-a_3$$
$$\frac{1}{4}K_3^{-4}=-(5a_2^3-5a_2a_3+a_4)$$

In general , for any p ∈ N , an expansion of K_n^p is as, [5,page183]

$$K_n^p=pa_n+\frac{p(p-1)}{2}D_n^2$$
$$+\frac{p!}{(p-3)!3!}D_n^3+\ldots+\frac{p!}{(p-n)!n!}D_n^n$$

where

$$D_n^p=D_n^p(a_2,a_3,\ldots)$$

and by [12]

$$D_n^m(a_1,a_2,\ldots a_n)=\sum_{m=1}^{\infty}\frac{m!(a_1)^{\mu_1}\ldots(a_n)^{\mu_n}}{\mu_1!\ldots\mu_n!}$$

while $a_1 = 1$, and the sum is taken over all non negative integers $\mu_1,\ldots,\mu_n$ satisfying

$$\mu_1+\mu_2+\ldots+\mu_n=m,$$
$$\mu_1+2\mu_2+\ldots+n\mu_n=n$$

It is clear that

$D_n^n(a_1,a_2,\ldots a_n)=a_1^n$ [7].

Theorem 2.1: If f(z) ∈ A satisfies (1.1), is in the class $B_{\Sigma}^{\psi,\beta}(\phi)$

If $a_k= 0$; $(2\le k\le n-1)$, then

$$|a_n|\le\frac{2\cos\beta}{\psi_n}, n\ge 4$$

where β ∈(-π/2,π/2).

Proof: Let f (z) ∈ A be as given in (1.1)

Therefore,

$$e^{i\beta}\frac{(f*\psi)(z)}{z}=e^{i\beta}\left[1+\sum_{n=2}^{\infty}a_n\psi_n z^{n-1}\right] \qquad (2.1)$$

and for its inverse map, $g=f^{-1}$, we have

$$e^{i\beta}\frac{(f*\psi)(w)}{w}=e^{i\beta}\left[1+\sum_{n=2}^{\infty}b_n\psi_n w^{n-1}\right]$$
$$=e^{i\beta}\left[1+\sum_{n=2}^{\infty}K_{n-1}^{-n}(a_2,a_3,\ldots)\psi_n w^{n-1}\right] \qquad (2.2)$$

On the other hand, since f ∈ $B_{\Sigma}^{\psi,\beta}(\phi)$ and $g=f^{-1}$ ∈ $B_{\Sigma}^{\psi,\beta}(\phi)$, by definition, there exist two Schwarz functions

$p(z) = c_1z+c_2z^2+c_3z^3+\ldots$ and

$q(w) = d_1w+d_2w^2+d_3w^3+\ldots$ such that

$$e^{i\beta}\left[1+\sum_{n=2}^{\infty}a_n\psi_n z^{n-1}\right]=\phi(p(z))$$
$$e^{i\beta}\left[1+\sum_{n=2}^{\infty}b_n\psi_n w^{n-1}\right]=\phi(q(w))$$

where $\phi(p(z))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\phi_k D_n^k(c_1,c_2,\ldots,c_n)z^n$ (2.3)

and $\phi(q(w))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\phi_k D_n^k(d_1,d_2,\ldots,d_n)w^n$ (2.4)

Comparing the corresponding coefficients of (2.1)and (2.3) we have

$$e^{i\beta}\left[1+\sum_{n=2}^{\infty}a_n\psi_n z^{n-1}\right]=$$

$$\sum_{k=1}^{n-1}\phi_k D_{n-1}^k(c_1,c_2,...,c_{n-1}), n\geq 2 \qquad (2.5)$$

Similarly from (2.2) and (2.4) we have

$$e^{i\beta}\left[1+\sum_{n=2}^{\infty}b_n\psi_n z^{n-1}\right]=$$

$$\sum_{k=1}^{n-1}\phi_k D_{n-1}^k(d_1,d_2,...,d_{n-1}), n\geq 2 \qquad (2.6)$$

Now, (2.5) and (2.6) for $a_k = 0$ $(2 \leq k \leq n-1)$, respectively, yield

$$a_n\psi_n e^{i\beta}=\phi_1 c_{n-1}\cos\beta \qquad (2.7)$$

$$\text{and} -a_n\psi_n e^{i\beta}=\phi_1 d_{n-1}\cos\beta \qquad (2.8)$$

Taking the absolute values of a_n in (2.7) or (2.8) and using the facts $|\phi_1| \leq 2$, $|c_{n-1}| \leq 1$ and $|d_{n-1}| \leq 1$ we obtain

$$|a_n|\leq\frac{2\cos\beta}{\psi_n} \qquad (2.9)$$

Our next theorem clearly shows the unpredictable behavior of the initial coefficients of classes of bi-univalent functions.

Theorem 2.2: If $f\in A$ satisfies (1.1), is in the class $B_{\Sigma}^{\psi,\beta}(\phi)$. Then

i) $|a_2|\leq\min\left\{\frac{2\cos\beta}{\psi_2},\sqrt{\frac{4\cos\beta}{\psi_3}}\right\}$

ii) $|a_3|\leq\min\left\{\frac{4\cos\beta}{\psi_3},\frac{2\cos\beta}{\psi_3}+\frac{4\cos^2\beta}{\psi_2^2}\right\}$

iii) $|2a_2^2-a_3|\leq\frac{4\cos\beta}{\psi_3}$

where $\beta\in(-\pi/2,\pi/2)$.

Proof: Letting n = 2 and n = 3 in (2.5) and (2.6) respectively, imply

$$a_2\psi_2 e^{i\beta}=\phi_1 c_1\cos\beta \qquad (2.10)$$

$$a_3\psi_3 e^{i\beta}=(\phi_1 c_2+\phi_2 c_1^2)\cos\beta \qquad (2.11)$$

$$-a_2\psi_2 e^{i\beta}=\phi_1 d_1\cos\beta \qquad (2.12)$$

$$(2a_2^2-a_3)\psi_3 e^{i\beta}=(\phi_1 d_2+\phi_2 d_1^2)\cos\beta \qquad (2.13)$$

From (2.10) or (2.12) we have

$$|a_2|=\left|\frac{\phi_1 c_1\cos\beta}{e^{i\beta}\psi_2}\right|\leq\frac{2\cos\beta}{\psi_2} \qquad (2.14)$$

From (2.10) and (2.12) we have

$$(2a_2^2)\psi_2^2 e^{2i\beta}=\phi_1^2(c_1^2+d_1^2)\cos^2\beta$$

$$a_2^2=\frac{\phi_1^2(c_1^2+d_1^2)}{2e^{2i\beta}\psi_2^2}\cos^2\beta \qquad (2.15)$$

From (2.11) and (2.13) we have

$$(2a_2^2)\psi_3 e^{i\beta}=\left[\phi_1(c_2+d_2)+\phi_2(c_1^2+d_1^2)\right]\cos\beta$$

$$a_2^2=\frac{\left[\phi_1(c_2+d_2)+\phi_2(c_1^2+d_1^2)\right]}{2\psi_3 e^{i\beta}}\cos\beta$$

$$a_2=\sqrt{\frac{\left[\phi_1(c_2+d_2)+\phi_2(c_1^2+d_1^2)\right]}{2\psi_3 e^{i\beta}}\cos\beta}$$

$$|a_2|\leq\sqrt{\frac{4\cos\beta}{\psi_3}}$$

and combining this with the inequality (2.14) we obtain the desired estimate on the coefficient $|a_2|$ as asserted in the theorem.

From (2.11) we have

$$|a_3|=\left|\frac{(\phi_1 c_2+\phi_2 c_1^2)\cos\beta}{e^{i\beta}\psi_3}\right|\leq\frac{4\cos\beta}{\psi_3} \qquad (2.16)$$

On the other hand, again from (2.11) and (2.13) we have

$$2(a_3-a_2^2)\psi_3 e^{i\beta}=$$

$$\left[\phi_1(c_2-d_2)+\phi_2(c_1^2-d_1^2)\right]\cos\beta$$

Since $c_1 = -d_1$, we have

$$a_3-a_2^2=\frac{\phi_1(c_2-d_2)}{2e^{i\beta}\psi_3}\cos\beta$$

and using (2.15)

$$a_3=\frac{\phi_1(c_2-d_2)}{2e^{i\beta}\psi_3}\cos\beta+\frac{\phi_1^2(c_1^2+d_1^2)}{2e^{2i\beta}\psi_2^2}\cos^2\beta$$

which implies $|a_3|\leq\frac{2\cos\beta}{\psi_3}+\frac{4\cos^2\beta}{\psi_2^2}$

and combining this with the inequality (2.15) we obtain the desired estimate on the coefficient $|a_3|$ as asserted in the theorem 2.2.

Also from (2.13) we have $|2a_2^2-a_3|\leq\frac{4\cos\beta}{\psi_3}$

Remark2.1: On taking special values for $\psi(z)$, we obtain a known result due to S.P.Goyal et al [20] and Srivastava et al[8].

References:

1. A.G.Alamoush and M.Darus "Coefficient bounds for new subclasses of bi-univalent functions using Hadamard product",Acta Universitatis Apulensis,no.38, 2014,pp.153-161.
2. *Shashi Bhushan, R. Karan Singh , Nazia Naqvi,* A Class of Cost Efficient Estimators of Population Ratio ; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 662-668
3. B.A.Frasin and M.K.Aouf, "New subclasses of bi-nivalentfunctions," AppliedMathematicsLetters,vol.24,no.9,2011,pp.156 9– 1573.
4. D.A. Brannan, J.G. Clunie (Eds.), "Aspects of Contemporary Complex Analysis" (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1 -20, 1979), Academic Press, New York and London, 1980.
5. E. Netanyahu, "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in |z|<1,Arch.Rational Mech.Anal.32,1969,pp.100-112.
6. H. Airault and A. Bouali, "Differential calculus on the Faber polynomials", Bull. Sci. math.vol.130,no.3,2006,pp.179–222.
7. H. Airault and J. Ren, "An algebra of differential operators and generating functions on the set of univalent functions", Bull. Sci. Math. vol.126 ,no.5,2002,pp. 343–367.
8. H. Airault, "Remarks on Faber polynomials," International Mathematical Forum. Journal for Theory and Applications, vol. 3,no.9-12,2008,pp.449–456.
9. Preety, Stability Result for S-Iteration in Convex Metric; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 355-357
10. H.M. Srivastava, A.K. Mishra and P. Gochhayat, "Certain subclasses of analytic and biunivalent functions", Appl. Math. Lett. 23 ,2010,pp.1188–1192.
11. J.M. Jahangiri and S.G. Hamidi, "Coefficient estimates for certain classes of bi-univalent functions", Int. J. Math. Sci.,Art. ID 190560,2013, 4 pp.
12. J.M. Jahangiri, S.G. Hamidi and S.A. Halim, "Coefficients of bi-univalent functionswith positive real part derivatives", Bull. Malaysian Math. Soc. (2) vol.37, no.3 ,2014,pp. 633–640.
13. 11.M. Lewin, "On a coefficient problem for bi-univalent functions", Proc. Amer. Math. Soc. 18 ,1967,pp. 63–68.
14. M.Lellis Thivagar, M.Arockia Dasan, G.Jayaparthasarathy, Remarks on Weakly Open Sets in Extended Bitopological; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 714-717
15. P.G. Todorov, "On the Faber polynomials of the univalent functions of class Σ", J. Math. Anal. Appl. vol.162,no.1,1991,pp. 268–276.
16. P.L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, 1983.
17. R. M. Ali, S. K. Lee, V.Ravichandran, and S. Supramaniam, "Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions," Applied Mathematics Letters, vol. 25, no. 3, 2012,pp.344–351.
18. *S. Sudha, V. Kanniga,* Superposition of Stars on Cycles and N-Centipedes; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 634-636
19. S. Porwal and M. Darus, "On a new subclass of bi-univalent functions", J. Egyptian Math. Soc. vol.21,no.3,2013,pp.190–193.
20. S. S. Ding, Y. Ling, and G. J. Bao, "Some properties of a class of analytic functions," Journal of Mathematical Analysis and Applications",vol.195,no.1,1995,pp.71–81.
21. S.G. Hamidi and J.M. Jahangiri, "Faber polynomial coefficient estimates for analytic bi-closeto-convex functions", C. R. Acad. Sci. Paris, Ser. I 352, 1 ,2014,pp.17–20.
22. Serap Bulut, "Faber Polynomial Coefficient Estimates For A Subclass Of Analytic Bi-Univalent Functions Defined By Salagean Differential Operator" Matematiqki Vesnik,vol.67, no.3,2015,pp. 185–193
23. S.G.Hamidi and .M.Jahangiri,"Faber polynomial estimates for biunivalent functions defined by subordination",Bull.Iranian.Math.Soc,vol41,No.5,2 015,pp.1103-1119.
24. S.P.Goyal and Pranay Goswami, "Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivates", Journal of the Egyptian Mathematical Society,Vol 20,2012,pp.179-182.

***

Karthiyayini.O/Sivasankari.V/
Department of Science and Humanities/PESIT Bangalore South Campus/
Hosur road/1km before Electronic City/Bengaluru -100.