A NOTE ON ZERO DIVISOR GRAPH OF POLYNOMIAL RING OVER Z

SHEELA SUTHAR , OM PRAKASH

Abstract: Let $\Gamma(Z_n[x])$ be the zero-divisor graph of the ring $Z_n[x]$. Two vertices f(x) and g(x) of $\Gamma(Z_n[x])$ are adjacent if and only if f(x)· g(x) = 0. We discuss the partiteness, clique number and chromatic number of $\Gamma(Z_n[x])$ and co-relate it with the prime factorization of n. We also find the cases in which $\Gamma(Z_n[x])$ is isomorphic to Z_n .

Keywords: Chromatic number, Clique of graph, Commutative ring, Perfect graph and zero-divisor graph.

Introduction: Throughout this paper $Z_n[x]$ is a commutative polynomial ring over Z_n . We use Z($Z_n[x]$) to denoted the set of zero divisors of polynomial ring $Z_n[x]$. The zero-divisor graph of $Z_n[x]$ is denoted by $\Gamma(Z_n[x])$ and it is the collection of non-zero zero-divisors of $Z_n[x]$. Any two vertices are adjacent if f(x)·g(x) = 0. The concept of the zero-divisors graph of a ring was first introduced in 1988 by I. Back in when discussing the coloring of a commutative ring. He considered all elements of ring were vertices of the graph. D. D. Anderson and Naseer used this same concept in [1]. We adopt the approach used by D. F. Anderson and Livingston in [2] and consider only non-zero zero-divisors as vertices of a graph.

In this paper we studied the properties (partiteness, clique number and chromatic number) of $\Gamma(Z_n[x])$. A simple graph G with n vertices is said to be a complete graph if every vertex is adjacent to all its other vertices and it is denoted by K_n . The vertex with minimum eccentricity in a graph is called its centre. A graph may have more than one centre. Let G be a graph with vertex set V(G) and edge set E(G). Then G is said to be complete p-partite if V(G) partitioned into p disjoint sets $V_1, V_2, \cdots, V_p$ such that no two vertices within any V_i are adjacent, but for every $v \in V_i$, $u \in V_j$, u and v are adjacent.

An induced complete sub-graph of any graph G is called its clique and the order of the largest clique in the graph is its clique number, denoted by $\omega(\Gamma(G))$. A proper coloring of a graph G is a function that assigns a color to each vertex such that no two adjacent vertices have the same color. The chromatic number of graph denoted by $\chi(\Gamma(G))$, is the minimum number of colors required for proper coloring. A graph G is said to be perfect if for every sub-graph H of a graph G, chromatic number is equal to the clique number; i.e. $\omega(H) = \chi(H)$. If a graph G does not contain P_4 as an induced sub-graph, then G is a perfect graph.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic, if there exists a one to one correspondence between their vertices and their edges, preserving their adjacency relationship. So two graph G_1 and G_2 are isomorphic, if there exist a mapping $f : G_1 \to G_2$, such that (i) f is one-one, (ii) f is onto and (iii) If two vertices $v_i, v_j \in V_1$ are connected by an edge $e \in E_1$, then their images $f(v_i)$ and $f(v_j) \in V_2$ should have an edge $f(e) \in E$.

Two polynomials of $Z_n[x]$ are said to be in Z($Z_n[x]$) if coefficients of these polynomials are non-zero zero-divisor in Z_n and degree of the resultant polynomial (which obtained by multiplying them) is equal to n.

For example: - $(10x^{20}) \cdot (5x^5) = 0$

Here f(x) =$10x^{20}$ is annihilated by a polynomial g(x) = $5x^5$ and degree of the resultant polynomial is 25.

Lemma: - $Z_p[x]$ have no vertices and no edges, where p is prime number.

Proof: - Since p is prime, so $Z_p[x]$ is an integral domain. Now, proof is obvious.

Theorem 1:- $Z_{pq}[x]$ is complete bi-partite graph and need only two color for coloring the graph so the chromatic number is two, when p, q both are distinct prime number.

Proof: - Let f(x) and g(x) be two elements of Z($Z_{pq}[x]$). If pq/ f(x)g(x), then f(x)·g(x) = 0. Also, any element of Z($Z_{pq}[x]$) is divisible by p is connected to an element divisible by q. It forms a bi-partite graph. It is also clear that two elements are not adjacent if both are divisible either p or q. Then, there is a clique of order two.

Let, if possible, there is a clique of order three. Then there exist a(x), b(x) and c(x) ∈ Z($Z_{pq}[x]$) such that a(x) · (x) = 0; b(x)·c(x) = 0 and a(x) · c(x) = 0. We have only two primes p and q and each element is divisible by one of them. If all elements are divisible by p, then condition is fail, it also fails if all are divisible by q. So without loss of generality, we can assume p / a(x) and q / b(x). This implies that p / c(x) by our assumption that b(x) · c(x) = 0. However, if both c(x) and a(x) are divisible by p and their product cannot be zero, which is a contradiction. Therefore, $\Gamma(Z_{pq}[x])$ cannot have a clique of order three and hence cannot have a clique of higher order. So $\omega(\Gamma(Z_{pq}[x])) = 2$ and therefore, $\Gamma(Z_{pq}[x])$ has chromatic number two.

Theorem 2: - If p is prime number, then $\Gamma(Z_{p^2}[x])$ is K_{p-1} graph and chromatic number is p-1.

Proof: - Let f(x) $\in$ Z$(Z_{p^2}[x])$. Then p/f(x). Also for any f(x) and g(x) $\in$ Z $(Z_{p^2}[x])$, p^2 / f(x)g(x), implies f(x) · g(x) = 0. So, in $\Gamma(Z_{p^2}[x])$, each vertex is adjacent to every other vertex. Further-more, there are (p-1) non-zero elements that are divisible by p, (not by p^2), but the connectivity is hold when multiplication of two elements is divisible by p^2. So, the graph is complete with (p-1) vertices and this is a special type of p-partite graph in which each of the p sets of vertices contains only one vertex. Graph is complete with (p-1) vertices and it must be required (p-1) colors for coloring the graph. Hence the chromatic number of graph is (p - 1).

Theorem 3: - The graph $\Gamma(Z_{p^3}[x])$ is complete p-partite and required p colors for coloring the graph, where p is prime.

Proof: - Let f(x) $\in$ Z$(Z_{p^3}[x])$. Then p/f(x) and for any f(x) $\neq$ 0, g(x) $\neq$ 0 in Z $(Z_{p^3}[x])$ such that f(x) · g(x) = 0, then p^2 / f(x) or p^2 / g(x) or both. In case, if p^2 / f(x) then f(x) · g(x) = 0, for all g(x)$\in (Z_{p^3}[x])$, i.e. f(x) is adjacent with all other vertices. In $\Gamma(Z_{p^3}[x])$, there are (p – 1) non-zero elements which are divisible by p^2 and also all the remaining elements of $\Gamma(Z_{p^3}[x])$, divisible by p, product of any two elements are not zero. So, there are no edges between in these elements which are divisible by p. Thus, we can split V($\Gamma(Z_{p^3}[x])$) into p independent sets $V_1, V_2, \cdots, V_p$ and each of V_i, $1 \le i \le p-1$ will contain only one vertex which is divisible by p^2 and V_p will contain those zero divisors which are divisible by p. Hence, $\Gamma(Z_{p^3}[x])$ is a complete graph. Here, we have p independent sets of vertices so we need p colors for coloring the graph.

Theorem 4: - $\Gamma(Z_{p^2q}[x])$ has an induced complete p-partite and an induced complete sub-graph, where p and q are distinct prime number. Chromatic number of this graph is p.

Proof: - In case of integer p^2q, the centre of graph will be ($\frac{p^2q}{2}$). Those vertices which are adjacent to the center will be divisible by q and pq. The centre is not adjacent to those vertices which are divisible by p. Therefore, the vertices which are adjacent to the centre are divisible by q and pq, so it makes a complete p-partite graph. On the other hand, if f(x) and g(x) are two non-zero zero-divisor in $\Gamma(Z_{p^2q}[x])$ and pq/f(x) and pq/g(x), then f(x) and g(x) are adjacent in $\Gamma(Z_{p^2q}[x])$. They are also adjacent to ($\frac{p^2q}{2}$ and divisible by p. All those zero divisors that are divisible by pq and ($\frac{p^2q}{2}$ or that are divisible pq and p, then it's make a clique which is an induced complete sub-graph. There are at most p elements in clique, so we can color all such zero divisors with k_p. If we have another zero divisor c(x), then either p X c(x) or q X c(x), otherwise c(x) would be in clique. If p X c(x) then c(x) is not adjacent to any vertex in the clique. So we can color it with k_1. If p/c(x), then c(x) is adjacent to every element of clique. So, we can color c(x) with k_p. If non-zero zero-divisors c(x) and d(x) are not in clique, then there is no adjacency between c(x) and d(x). Hence we have required single color for both. Thus, we required exactly p colors for coloring the graph. So, the chromatic number of the graph is p.

Theorem 5: - $\Gamma(Z_{pqr}[x])$ has an induced complete p-partite graph. If p, q, r all are distinct prime then graph is three colorable and chromatic number is three.

Proof: - In case of Z_{pqr}, the graph will contain divisor of p but not the divisor of q. Here, (pqr/2) is the centre of the graph and it is adjacent with all divisors of p but not q and r, then there are no edges between divisors of p, q and r, the divisors of r are adjacent with all divisors of pq and are not adjacent with divisors of q and divisors of pr are adjacent to divisors of q. These two conditions will form an induced complete graph. For every zero divisor f(x), we must have that at least one of p, q, r does not divide f(x), since if all of these primes divide f(x), then f(x) is divisible by pqr then it becomes zero divisor. Thus p_i X f(x), where i is minimal, we color f(x) with K_i. Using this scheme, if f(x) and g(x have the same color, say K_j, then j X f(x) or j X g(x), and thus f(x) · g(x) ≠ 0. Hence f(x) and g(x) cannot be adjacent. Therefore, we have three primes pr, qr, pq that form a clique of order three and we done proper coloring with three colors. So, chromatic number is three.

Theorem 6: - $\Gamma(Z_{p^4}[x])$ have an induced complete sub-graph and a complete p-partite graph, where p is prime.

Proof: - In $Z(Z_{p^4}[x])$ there are p-1 elements divisible by p^3. Elements divisible by p^3 are adjacent with all elements divisible by p but are not adjacent with the elements which are divisible by p^2 and its form a complete p-partite graph. The remaining divisors of p^2 are adjacent each other and its form a clique. This clique is an induced complete sub-graph.

For example $\Gamma(Z_{16})$ and $\Gamma(Z_{81})$ has induced complete sub-graph and a complete p-partite graph.

In general, if we take p as power of n which is either odd or even, then we get a graph with an induced complete p-partite graph and an induced complete sub-graph. This type of graphs has chromatic number $p^{\frac{n}{2}}-1$ for n is even and $p^{\frac{n-1}{2}}$ for n is odd respectively. In case of product of distinct primes, we get complete p-partite graph. For proper coloring of this p-partite graph we required color according to p-partite, if we take product of two distinct prime numbers then the chromatic number are two and if we take product of three distinct prime numbers then the chromatic number are three and so on. In general if we take $m = p_1, p_2, \cdots, p_n$ where each p is a distinct prime. Then $\Gamma(Z_n[x])$ is n colorable.

According to Daniel Endean, Kristin Henry and Erin Manlove [6], the graph $\Gamma(Z_n)$ is perfect if and only if n = p^k for some prime p and n = $p_1 p_2$ for distinct primes p_1, p_2.

Now we show behavior of both graphs $\Gamma(Z_n)$ and $\Gamma(Z_n[x])$ are same. According to Daniel En-dean [5], our graph $\Gamma(Z_n[x])$ is also perfect in both cases. And graph $\Gamma(Z_n[x])$ is also isomorphic to $\Gamma(Z_n)$, when n = p^m, where p is any prime and m is even or odd. Now we show that graph is isomorphic in other cases when graph is not perfect. Here, we would like to state that in the graph $\Gamma(Z_n[x])$, we have blocks of graph which are not connected to each other and each block is equivalent to Z_n.

Theorem 7: - $\Gamma(Z_n[x])$ is isomorphic to $\Gamma(Z_n)$, for *n* = *pq*, *pqr*, $p^m q$, where *m* is a positive integer and *p*, *q*, *r* are distinct primes.

Proof: - Let for each $a \in Z(Z_n)$ we have uniquely expression for *a* as *a* = $a_k p^k + a_{k+1} p^{k+1} + \cdots + a_{n-1} p^{n-1}$ where $k < n$ and $0 < a_i < p$ for each

$i = k, \cdots\cdots, n \geq 1$. We define a mapping ϕ from $Z(Z_n)$ to $Z(Z_n[x])$ by $\phi_p(a)$ = a(x) = $a_k x^k + a_{k+1} x^{k+1} + \cdots + a_{n-1} x^{n-1}$. Suppose *a* and *b* are equivalent in $Z(Z_n)$, then *a* = *b* + *qp* for some $q \in Z$. Now *a(x)* = *b(x)* + *q(* x_n *)* and we see that $\phi_p(a)$ is equivalent to $\phi_p(b)$ in $Z(Z_n[x])$ according defined mapping. Hence mapping is well defined. Let for any $a(x) \in Z(Z_n[x])$ we know a(x) = $a_k x^k + a_{k+1} x^{k+1} + \cdots + a_{n-1} x^{n-1}$ where 0 < a_i < p for all $i = 1,2,\cdots,n$ and $a_i \neq 0$ for at least one *i*. Clearly, $a \in Z(Z_n)$ and hence *pq*/*a* (for all above mention cases in theorem). Hence $\phi_p(a)$ = a(x), and ϕ_p is onto. By our consideration and [6] it must also be one-to-one. At last we will show that ϕ_p preserves adjacency. Let for any non-zero zero divisors of $a, b \in Z(Z_n)$, they are adjacent each other if a · b = 0 and for f(x) · g(x) = 0 where f(x) · g(x) $\in Z(Z_n[x])$ is also adjacent because the coefficients of polynomials are belongs to . $Z(Z_n)$ Then multiplication of polynomials are zero if multiplication of coefficients is zero and also degree of resultant polynomials is equal to n, where n is any prime number. By defined mapping ϕ_p is preserving the adjacency. Then $\Gamma(Z_n[x])$ is isomorphic to $\Gamma(Z_n)$.

References:

1. D. D. Anderson and M. Naseer, Beck's coloring of Commutative Ring, J. Algebra, 159 (2) (1993), 500-514.
2. D. F. Anderson and P. S. Livingston, The zero divisor graph of a Commutative ring, J. Algebra, 17 (2) (1999), 434-447.
3. I. Beck, Coloring of Commutative rings, J. Algebra, 16 (1) (1988), 208-226.
4. A. Duane, A proper Colorings and p-Partite structures of zero divisor graph, Rose-Hulman Undergraduate Math Journal, 7 (2) (2006).
5. *Jisha Anna Alex, Sheena Anees,* A Secure Single Sign-on Mechanism With Data Encryption; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 392-397
6. D. Endean, K. Henry and E. Manlove, Zero divisor graphs of Zn and Polynomial Quotient Rings over Zn, Rose-Hulman Undergraduate Math Journal, 8 (2) (2007).
7. N. H. McCoy, Remarks on divisors of zero, American Mathematical Monthly, 49 (1942),286-295.

Sheela Suthar , Dept of Mathematics and Statistics, Banasthali, Rajasthan- 304022.(India)
Om Prakash , Dept of Mathematics, IIT Patna Patliputra colony,Patna-, 800013.