A METHOD FOR APPROXIMATING SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS BY USING SHEPARD KERNAL BASED FUZZY TRANSFORMS

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Abstract: Fuzzy transforms developed by Irina Perfilieva is a novel, mathematically well founded soft computing tool with many applications. These techniques are based on mainly two transforms, direct fuzzy transform and inverse fuzzy transform. B. Bede and I. J. Rudas introduced Shepard kernel based fuzzy transform but its applications are not elaborately done. In this paper we develop an approximating model based on Shepard kernel based fuzzy transform and apply the model for numerical solution of ordinary differential equations.

Keywords: Fuzzy partitions, Fuzzy Transforms, Shepard Kernals, Shepard Kernal based Fuzzy Transform, Ordinary Differential Equations.

Introduction: In classical Mathematics, various types of transforms are introduced (e.g. Laplace transform, Fourier transform, wavelet transform etc.) by various researchers. In 2001 Irina Perfilieva introduced fuzzy transform in her paper [3]. Latter on fuzzy transform is applied in to various fields, like image processing, data mining etc in the papers [5, 10]. The fuzzy transform provides a relation between the space of continuous functions defined on a bounded domain of real line R and R^n .Similarly inverse fuzzy transform identified each vector of R^n with a continuous map. The central idea of the fuzzy transform is to partition the domain of the function by fuzzy sets.

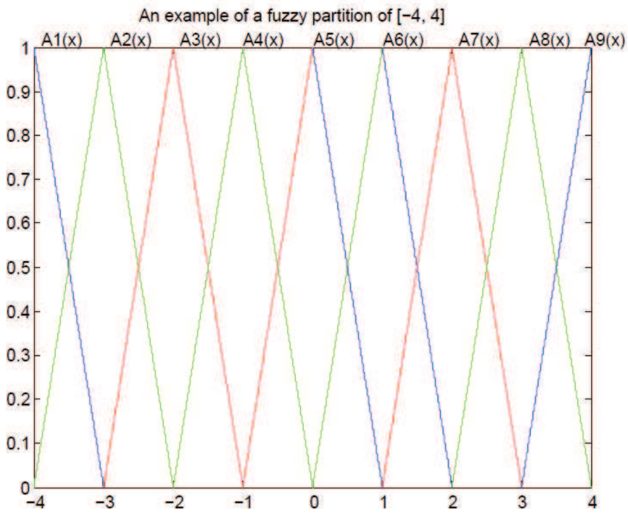
Definition 1.1([6]): Let $[a, b]$ be an interval of real numbers and $x_1 < x_2 < \ldots < x_n$ be fixed nodes within $[a, b]$ such that $x_1 = a$, $x_n = b$ and $n \geq 2$ .We say that fuzzy sets $A_1, A_2, \ldots, A_n$ identified with their membership functions $A_1(x), A_2(x), \ldots\ldots, A_n(x)$ and defined on $[a, b]$ form a fuzzy partition of $[a, b]$ if they fulfill the following conditions for $i = 1, 2, \ldots, n$.

1. $A_i : [a , b] \to [0 , 1]$, $A_i(x_i) = 1$.
2. $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$
3. $A_i(x)$ is continuous.
4. $A_i(x)$ is monotonically increasing on $[x_{i-1}, x_i]$ and monotonically decreasing on, $[x_i, x_{i+1}]$
5. $\sum_{i=1}^{n} A_i(x) = 1$, for all x.
6. $A_i(x_i - x) = A_i(x_i + x)$, for all $x \in [0, h]$, $i = 2,\ldots,n-1$, $n > 2$.
7. $A_{i+1}(x) = A_i(x - h)$, for all $x \in [a + h, b]$, for $i = 2, 3,\ldots,n-2$, $n > 2$.

Where h is the uniform distance between two nodes.

The shape of basic functions is not predetermined and therefore it can be chosen additionally according to further requirements.

The following figure shows a fuzzy partition of the interval $[-4, 4]$, with triangular membership functions.



The following expression gives the formal representation of such triangular membership functions.

$$A_1(x) = \begin{cases} -3 - x, x \in [x_1, x_2] \\ 0, \quad otherwise \end{cases}$$

and for $i = 2, 3, \ldots, n-1$.

$$A_i(x) = \begin{cases} x - x_{i-1}, & if\ x \in [x_{i-1}, x_i] \\ 1 - x + x_i, & if\ x \in [x_i, x_{i+1}] \\ 0, & otherwise \end{cases}$$

$$A_n(x) = \begin{cases} x - x_{n-1}, & if\ x \in [x_{n-1}, x_n] \\ 0, & otherwise \end{cases}$$

Fuzzy Transform: In this section we first give the definition of fuzzy transform given by Irina Perfilieva in 2006.

Definition 2.1([6]): Let $f(x)$ be a continuous function on $[a , b]$ and $A_1(x), A_2(x),\ldots,A_n(x)$ be basis functions determining a uniform fuzzy partition of $[a , b]$ .The n-tuple of real numbers $[F_1, F_2, \ldots, F_n]$ such that

$$F_i = \frac{\int_a^b f(x) A_i(x) dx}{\int_a^b A_i(x) dx}, \quad i = 1, 2, \ldots, n. \qquad (1)$$

will be called the F- transform of f w.r.t. the given basis functions. Real's F_i are called components of the F-transform.

Lemma 2.1([5]): Let f be any continuous function defined on [a, b],but function f is twice continuously differentiable in (a, b) and $A_1(x)$, $A_2(x)$,…,$A_n(x)$ be basis functions determining a uniform fuzzy partition of [a ,b]. Then for each $i = 1, 2, \dots, n$, $F_i = f(x_i) + O(h^2)$

Proof: Perfilieva (2004).

Now a question arises in the minds that can we get back the original function by its fuzzy transform. The answer is we can reconstruct an approximate function to the original function. For that purpose Perfilieva define inverse fuzzy transform.

Definition 2.2([6]): Let A_1, A_2, …, A_n be basic functions which form a uniform fuzzy partition of [a, b] and f be a function from $c([a, b])$. Let $F_n[f]$ = [F_1, F_2, …, F_n] be the fuzzy transform of f with respect to A_1, A_2, …, A_n . Then the function defined by

$f_{F,n}(x) = \sum_{i=1}^{n} F_i.A_i(x)$ is called the inverse fuzzy transform of f with respect to A_1, A_2, …, A_n .

The following theorem shows that the inverse fuzzy transform can approximate the original continuous function f with a very small precision.

Theorem2.1([6]): Let f be a continuous functions defined on [a, b]. Then for any ε > 0 there exist n_ε and a uniform fuzzy partition $A_1, A_2, \dots, A_{n_\varepsilon}$ of [a, b] such that for all $x \in [a, b]$, $|f(x) - f_{F,n_\varepsilon}(x)| \le \varepsilon$

Shepard Kernels based F-transform: Shepard Kernals are well known functions in the field of numerical analysis. In the paper "Approximation properties of fuzzy transform", by B. Bede and I.J. Rudas [2], a new type of fuzzy transform has been introduced by using Shepard kernels as basic functions, and named these transform as Shepard kernels based fuzzy transform. In this section we apply these new transform for approximating continuous function and study its applications for compression and decompression of gray image.

The Shepard kernels basic functions are defined as follows:

Definition 3.1([2]): Consider the closed interval$[a, b]$, and a partition of these intervals $\{y_1, y_2, \cdots, y_k\}$, then Shepard Kernels A_i for $i = 1,2,\cdots,k$ is defined as

$$A_i(x) = \begin{cases} \dfrac{|x - y_i|^{-\lambda}}{\sum_{j=0}^{k} |x - y_j|^{-\lambda}}, & if\ x \in [a, b] \backslash \{y_1, y_2, \cdots, y_k\} \\ \delta_{ij} & if\ x \in \{y_1, y_2, \cdots, y_k\} \end{cases}$$

Where δ_{ij} is Kronecker's delta and λ is a parameter.

One can check that the range of these Shepard Kernals functions is [0, 1]. Thus we can say that Shepard Kernels are fuzzy set defined on a closed interval. Also from the definition of the Shepard Kernels it can be shown that $\sum_{i=1}^{k} A_i(x) = 1$. These Shepard Kernels are continuous functions. Thus we can say that these Shepard Kernels satisfies the important properties of the fuzzy partition of closed intervals defined in Definition1.1 except the support of the fuzzy set and B. Bede and I. J. Rudas shows that for proving the main theorem of fuzzy transform these property is not needed.

Definition 3.2([2]): Let [a, b] be any closed interval and f(x) be any real valued continuous functions defined on [a, b] . Let A_i for $i = 1,2,\cdots,k$ are the Shepard kernels defined on [a, b]. Then the n-tuple of real numbers [$F_1, F_2, \dots, F_k$] such that

$F_i = \dfrac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx}$, $i = 1, 2, \dots, k$. is called the Shepard kernels based F- transform of f . Real's F_i are called components of the Shepard kernels based F-transform.

Definition 3.3([2]): Let A_1, A_2, …, A_k be basic functions which form a fuzzy partition of [a ,b] and f be a function from C([a, b]). Let $F_k[f]$ = [F_1, F_2, …, F_k] be the Shepard kernel based fuzzy transform of f with respect to A_1, A_2, …, A_k . Then the function

$f_{F,k}(x) = \sum_{i=1}^{k} F_i.A_i(x)$

is called the inverse Shepard kernel based fuzzy transform.

Now we state the main important theorem of this section.

Theorem 3.4: Let f be a continuous functions defined on [a, b]. Then for any ε > 0 there exist n_ε and a Shepard kernel based fuzzy partition A_1, A_2, …, A_k of [a, b] such that for all $x \in$ [a ,b],

$|f(x) - f_{F,k}(x)| \le \varepsilon$.

The proof of the above theorem is similar to the proof of Theorem 5.2 in [2].

Example 3.5: Consider the closed interval [0, 4] and let f be a function defined on [0, 4] as $f(x) = x^2 + 1$. We approximate these functions by using Shepard kernel based fuzzy transform. We consider a partition {0, .5, 1, 1.5, 2, 2.5, 3, 3.5, 4}. Then the graph of the Shepard kernel basic functions taken λ=3 is given below:

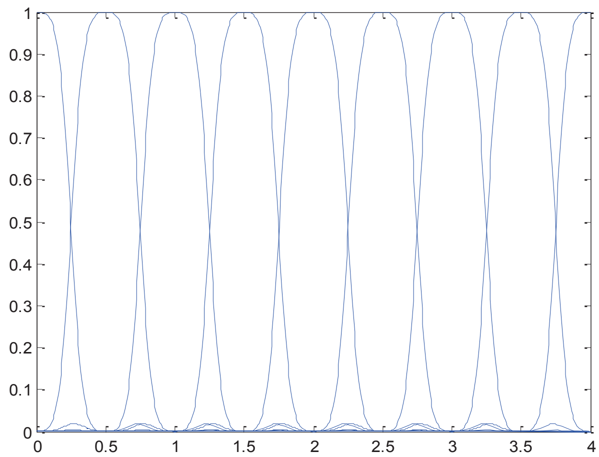


Figure 3.1: Shepard kernel based fuzzy partition of the closed interval [0, 4]

Now we have to evaluate the Shepard kernel based F-transform components of the function $f(x) = x^2 + 1$, with respect to the above Shepard kernels fuzzy partition. We evaluate these components by using software MATLAB and the corresponding values are

$$F_1 = 1.0484, F_2 = 1.2958, F_3 = 2.0442,$$
$$F_4 = 3.2889, F_5 = 5.0442, F_6 = 7.2904,$$
$$F_7 = 10.0154, F_8 = 13.2143, F_9 = 16.8166$$

Now by using Definition3.3 we evaluate the inverse Shepard kernel based fuzzy transform and the graph of these approximated curve and the original curve is given below, where red graph is the original curve and blue graph is our approximated curve:

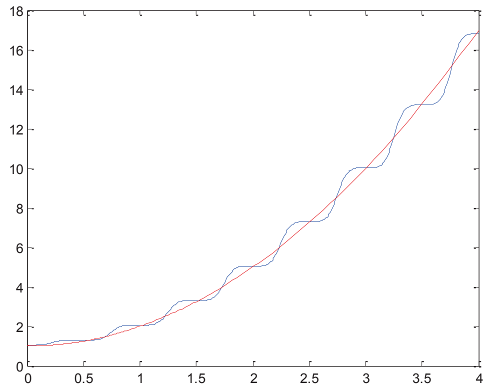


Figure 3.2: Graph of the exact function and the graph of the approximated function

From the above figure we can see that the Shepard kernel based fuzzy transform approximates continuous functions very much accurately.

Solution of ordinary Differential equation by using Shepard kernel based F-transform: Consider the differential of the following form

$$y''(x) = f(x, y) \tag{2}$$

$y'(x_1) = c$, $y(x_1) = d$ *where* f is any arbitrary function of x and y, and here we show that this differential equation can be solved by using Shepard kernel based fuzzy transform.

For solving the above equation we need a uniform Shepard kernel based fuzzy partition of the domain. Let, $a = x_1 < x_2 < \cdots < x_n = b$ be fixed nodes within the domain and consider the Shepard kernel based fuzzy partition, $A_1, A_2, \ldots, A_n$ defined on this domain. Here we also assume that all the node points are equidistant, i.e. $x_i - x_{i-1} = h$ (say).

Now we approximate $y'(x)$ and $y''(x)$ by the following formula:

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \tag{3}$$

$$y''(x) = \frac{y(x-h) - 2y(x) + y(x+h)}{h^2} + O(h^2) \tag{4}$$

Denote $y_1(x) = y(x+h)$ as a new function and $y_2(x) = y(x-h)$ as another new function.

Now if we apply the Shepard kernel based F-transform on both sides of equation (4), then by using these new functions and by linearity of Shepard kernel based F-transform, we obtain the relation between the Shepard kernel based fuzzy transform components of y, y_1, y_2 and y'' as follows

$$F_n[y''] = \frac{F_n[y_1] - 2F_n[y] + F_n[y_2]}{h^2} \tag{5}$$

where, $F_n[y''] = [Y_2'', Y_3'', \ldots, Y_{n-1}'']$,

$F_n[y_1] = [Y1_2, Y1_3, \ldots, Y1_{n-1}]$

$F_n[y_2] = [Y2_2, Y2_3, \ldots, Y2_{n-1}]$, and

$F_n[y] = [Y_2, Y_3, \ldots, Y_{n-1}]$, are the Shepard kernel based fuzzy transform components of y'', y_1, y_2 and y respectively. Note that these vectors are two components shorter since y_1 may not be defined on $[x_{n-1}, x_n]$ and y_2 may not be defined on $[x_1, x_2]$.

Now by using the definition of Shepard kernel based fuzzy transform it can be easily proved that, $Y1_k = Y_{k+1}$ and $Y2_k = Y_{k-1}$,for $k = 2, 3, \ldots, n-1$.

Indeed, for values of $k = 2, 3, \ldots, n-2$,

$$Y1_k = \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} y(x+h)A_k(x)dx = \frac{1}{h}\int_{x_k}^{x_{k+2}} y(t)A_{k+1}(t)dt = Y_{k+1}$$

For, $k = n-1$, the proof is similar. The proof of $Y2_k = Y_{k-1}$, for $k = 2, 3, \ldots, n-1$ is analogous. Therefore, we can write the components of the Shepard kernel based F-transform of y'' via components of the F-transform of y. So, we can write the equation (4) component wise as

$$Y_k'' = \frac{Y_{k+1} - 2Y_k + Y_{k-1}}{h^2}, \text{ for } k = 2, 3, \ldots, n-1.$$

Now, for $k = 2, 3, \ldots, n-1$, we introduce the following system of linear equations.

$$Y_2'' = \frac{Y_3 - 2Y_2 + Y_1}{h^2}$$
$$Y_3'' = \frac{Y_4 - 2Y_3 + Y_2}{h^2}$$
$$\vdots$$
$$Y_{n-1}'' = \frac{Y_n - 2Y_{n-1} + Y_{n-2}}{h^2}$$

The above system can be written in matrix form as

$$[Y''_2, \ldots, Y''_{n-1}]^T = D\,[Y_1, \ldots, Y_n]^T \tag{6}$$

where, D is the $(n-2) \times n$ matrix, given by

$$D = \frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & & \cdots & 1 & -2 & 1 \end{bmatrix}$$

Now, by using equation (6) and equation (2), we can write

$$D\,[Y_1, \ldots, Y_n]^T = [F_2, \ldots, F_{n-1}]^T$$

where, $F_2, \ldots, F_{n-1}$ are the corresponding Shepard kernel based fuzzy transform components of $f(x, y)$.

Now we use the initial conditions and make the matrix D as $n \times n$ matrix.

The initial conditions are given as,

$y(x_1) = c$, $\Rightarrow Y_1 = c$

and $y'(x_1) = d$, $\Rightarrow y(x_2) - y(x_1) = dh \Rightarrow Y_2 - Y_1 = dh$

By using the above initial conditions, we make the matrix D as square matrix of order $n \times n$ and also write the system of linear equations as

$$D^c\,[Y_1, \ldots, Y_n]^T = \left[\frac{c}{h^2}, \frac{d}{h}, F_2, \ldots, F_{n-1}\right]^T \tag{7}$$

Where, $D^c = \frac{1}{h^2}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 1-2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1-2 & 1 & 0 & \ldots 0 & 0 \\ \vdots & & & & & \\ 0 & 0 \ldots & & & 1-2 & 1 \end{bmatrix}$

Now, we solve the system of linear equation (7) by using any numerical techniques. Note here that the solution must exist since D^c is a invertible matrix.

Note 4.1: Here for solving both the systems of linear equations we need the value of F_i, which is Shepard kernel based fuzzy transform components of the function $f(x,y)$, with respect to x, therefore F_i will be dependent on y also. For overcoming these difficulties we approximate F_i as

$$F_i' = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x,Y_k)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx}$$

That means here we have assumed the value of the function y as constant for the interval (x_{k-1}, x_{k+1}). Here we omit the proof that $F_i - F_i' = O(h^2)$.

Now after evaluating all the fuzzy transform components of y, we will find an approximation of the function y by using inverse fuzzy transform.

For illustration of the above method we give the following example.

Example.4.1: Consider the following initial value problem

$$y''(x) = y + 1 + x + x^2, \ 0 \le x \le 4$$
$$y(0) = 8, \quad y'(0) = -2$$

For solving the above equation we take the Shepard kernel based fuzzy partition of [0, 4], as given in Example3.5. We take h= .5, thus the system of linear equation (7) can be written as

$$D^c\,[Y_1, \ldots, Y_9]^T = \left[\frac{8}{0.25}, \frac{-2}{.5}, F_2, \ldots, F_8\right]^T$$

Where, $D^c = \frac{1}{h^2}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 1-2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1-2 & 1 & 0 & \ldots 0 & 0 \\ \vdots & & & & & \\ 0 & 0 \ldots & & & 1-2 & 1 \end{bmatrix}$

Thus the above system of equation can be written as,

$$Y_1 = 8, \ Y_2 = 7, \quad Y_3 = 6 + 0.25F_2$$
$$Y_4 = 5 + .50F_2 + .25F_3$$
$$Y_5 = 5 + .75F_2 + .50F_3 + .25F_4$$
$$Y_6 = 5 + F_2 + .75F_3 + .50F_4 + .25F_5$$
$$Y_7 = 5 + 1.25F_2 + F_3 + .75F_4 + .50F_5 + .25F_6$$
$$Y_8 = 5 + 1.5F_2 + 1.25F_3 + F_4 + .75F_5 + .50F_6 + .25F_7$$
$$Y_9 = 5 + 1.75F_2 + 1.5F_3 + 1.25F_4 + F_5 + .75F_6 + .50F_7 + .25F_8$$

Now, for finding Y_3, Y_4, Y_5, Y_6, Y_7, Y_8 and Y_9, we need the value of F_2, F_3, F_4, F_5, F_6, F_7 and F_8, which are the 2nd, 3rd and 4th, 5th, 6th, 7th and 8th Shepard kernel based fuzzy transform components of the function $f(x,y) = y + 1 + x + x^2$. Now by using the Note4.1, we evaluate F_2, F_3, F_4, F_5, F_6, F_7 and F_8 by using software "MATLAB" and then subsequently we find Y_1 = 8, Y_2 = 7, Y_3 = 8.20, Y_4 = 12.212, Y_5 = 21.470, Y_6 = 37.857, Y_7 = 56.161, Y_8 = 101.9225 and Y_9 = 147.

Now by using these components we evaluate inverse Shepard kernel based fuzzy transform which is an approximation of the original solution and the graph of the original solution and our approximated solution is given below:

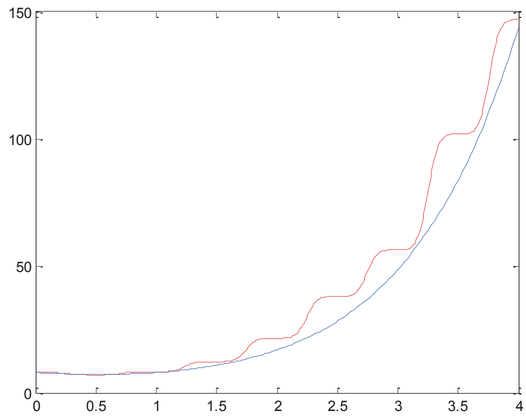


Figure 4.1: Graph of the exact solution of the differential equation and approximated solution by Shepard kernel based fuzzy transform

Conclusion: In this chapter we have introduced a new technique for solving 2nd order differential equation by using fuzzy transform and Shepard kernel based fuzzy transform and suitable examples are also given.

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