SOME APPLICATIONS OF EXTENDED q- DIFFERENCE OPERATORS IN NUMERICAL METHODS

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Abstract: In this paper, we define the some finite and infinite series of arithmetic progression in the field of Numerical Methods using the inverse of extended q-difference operators.

Key words: Extended q-difference operator, Finite Series, Infinite Series.

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Introduction: The theory of q-difference equations is based on the operator $D_q f(x)$ is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

Also, studies on linear q-difference equations were started at the beginning of the last century with the intensive works by Jackson [6], Carmichael [4], Mason [13], C.R.Adams [1], Trjitzinsky [16] and some others such as, Picared and Ramanujan. However, from 1930's up to the begining of 1980's, the theory of linear q-difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. Since 1980's an extensive and somewhat surprising interest in the subject reappeared in many areas of Mathematics, Physics and applications including new finite difference calculus and orthogonal polynomials, q-Combinatories, q-Arithmetics, integrable systems and variational q-Calculus [2, 17, 18].

In 2006, M.S.Manuel, et.al., extended from the difference operator to generalized difference operator is denoted by Δ_ℓ is defined on the real valued function $u(k)$ by

$\Delta_\ell u(k) = u(k+\ell) - u(k),\ k \in [0,\infty),\ \ell \in (0,\infty)$ and

developed the theory of difference equations in a different direction. Also, we obtained some interesting results in the field of Numerical Methods. By extending, the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and web like were studies for the solutions of difference equations involving Δ_ℓ. The results obtained can be found [8-12].

With this background, in this paper, we develop the basic theory for the extended q-difference operator $\Delta_{q(\ell)}$ and we define the inverse of q-difference operator $\Delta^{-1}_{q(\ell)}$ and obtain the formula for finding the sum of the higher powers of an arithmetic progression. Suitable examples are presented to establish the results in the field of Numerical Methods.

Preliminaries: In this section, we present some basic definitions and preliminary results for further subsequent discussions.

Definition 2.1. If u(k) is real valued function, then we define the extended q-difference operator $\Delta_{q(\ell)}$ as $\Delta_{q(\ell)} u(k) = u((k+\ell)q) - u(k),\ q \in (0,\infty).$ (1)

Lemma 2.2. The relation between $\Delta_{q(\ell)}$ and $E^{q(\ell)}$ is

$$E^{q(\ell)} = \Delta_{q(\ell)} + 1. \qquad (2)$$

Proof. The shift operator $E^{q(\ell)}$ is defined by

$$E^{q(\ell)} u(k) = u((k+\ell)q),\ k \in [0,\infty). \qquad (3)$$

The proof follows from the equations (1) and (3).

Lemma 2.3. If $q, \ell \in N(1) = \{1, 2, \ldots\}$, then

$$1 + \Delta_{q(\ell)} = (1+\Delta)^{q(\ell)}. \qquad (4)$$

Lemma 2.4. Let $u(k)$ and $v(k)$ be any two real valued functions. Then

$$\Delta_{q(\ell)}[u(k)v(k)] = v((k+\ell)q)\Delta_{q(\ell)}u(k) + u(k)\Delta_{q(\ell)}v(k).$$

Lemma 2.5. If $u(k)$ and $v(k) \neq 0$ are any two real valued functions, then

$$\Delta_{q(\ell)}\left[\frac{u(k)}{v(k)}\right] = \frac{v(k)\Delta_{q(\ell)}u(k) - u(k)\Delta_{q(\ell)}v(k)}{v(k)v((k+\ell)q)}.$$

Definition 2.6. The inverse of extended q-difference operator denoted by $\Delta^{-1}_{q(\ell)}$ is defined as if

$$\Delta_{q(\ell)} v(k) = u(k) \text{ then } v(k) = \Delta^{-1}_{q(\ell)} u(k) + c_j \qquad (5)$$

and the n^{th} order inverse operator denoted by $\Delta^{-n}_{q(\ell)}$ is defined as if

$\Delta^{n}_{q(\ell)} v(k) = u(k)$ then $v(k) = \Delta^{-n}_{q(\ell)} u(k) + c_j$,

where c_j is a constant.

Main Results And Its Applications:

Theorem 3.1: If k, ℓ and q are positive real values, then

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]} u\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right) = \Delta^{-1}_{q(\ell)} u(k) - \Delta^{-1}_{q(\ell)} u\left(\frac{k-\ell\sum_{t=1}^{\left[\frac{k}{\ell}\right]} q^t}{q^{\left[\frac{k}{\ell}\right]}}\right). \quad (6)$$

Proof. The proof follows from (5) and the relation

$$\Delta_{q(\ell)}\left[\sum_{r=0}^{\left[\frac{k}{\ell}\right]} u\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right)\right] = u(k).$$

Theorem 3.2. If $\lim_{k\to\infty} u(k) = 0$, then

$$\sum_{r=0}^{\infty} u\left(k\,q^r + \sum_{i=0}^{r} \ell\, q^i\right) = -\Delta^{-1}_{q(\ell)}\, u(k). \qquad (7)$$

Proof. (7) follows by (2).

Theorem 3.3. If k, ℓ and q are positive real values, then

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\left[\frac{(1-q^n)}{\left[\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right)+\ell\right]^n q^n} - \frac{\sum_{i=1}^{n} nC_i\, \ell^i \left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right)^{n-i}}{\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right)^n \left[\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right)+\ell\right]^n}\right]$$

$$= \frac{1}{k^n} - \frac{q^{\left[\frac{k}{\ell}\right]}}{\left[k-\ell\sum_{t=1}^{\left[\frac{k}{\ell}\right]} q^t\right]}. \qquad (8)$$

Proof. From (1) and (5), we have

$$\Delta^{-1}_{q(\ell)}\left[\frac{(1-q^n)}{[k+\ell]^n q^n} - \frac{\sum_{i=1}^{n} nC_i\, \ell^i\, k^{n-i}}{k^n [k+\ell]^n}\right] = \frac{1}{k^n}.$$

The proof follows from (6) and the above relation.

Theorem 3.4. If k, ℓ and $q \neq 1$ are positive real values, then

$$\sum_{r=0}^{\infty}\left[\frac{(1-q)}{q\left(k\,q^r + \ell\sum_{i=o}^{r} q^i + \ell\right)} - \frac{\ell}{\left(k\,q^r + \ell\sum_{i=o}^{r} q^i + \ell\right)^{(2)}_{\ell}}\right] = -\frac{1}{k}. \quad (9)$$

Proof. Taking $u(k) = \frac{1}{k}$ in (1), we have

$$\Delta_{q(\ell)}\left[\frac{1}{k}\right] = \frac{(1-q)k - q\ell}{k\left(k+\ell\right)q}.$$

From (5), we have

$$\Delta^{-1}_{q(\ell)}\left[\frac{(1-q)k - q\ell}{k\left(k+\ell\right)q}\right] = \frac{1}{k}.$$

The proof follows from (7) and the above relation.

The following example is the illustration of Theorem 3.3.

Example 3.5: In (8), by taking $n = 2, k = 43, \ell = 3$ and $\left[\frac{k}{\ell}\right] = 14$, we get

$$\sum_{r=1}^{14}\left[\frac{(1-q^2)\left[\frac{43-3\sum_{t=1}^{r} q^t}{q^r}\right]^2 - 6q^2\left[\frac{43-3\sum_{t=1}^{r} q^t}{q^r}\right] - 9q^2}{\left[\frac{43-3\sum_{t=1}^{r} q^t}{q^r}\right]^2\left[\frac{43-3\sum_{t=1}^{r} q^t}{q^r}+3\right] q^2}\right]$$

$$= \frac{1}{(43)^2} - \frac{q^{14}}{\left[43-3\sum_{t=1}^{14} q^t\right]}.$$

In Particular, when $q = 2$, we have

$$\sum_{r=1}^{14}\left[\frac{(1-2^2)\left[\frac{43-3\sum_{t=1}^{r} 2^t}{2^r}\right]^2 - 24\left[\frac{43-\ell\sum_{t=1}^{r} 2^t}{q^r}\right] - 36}{\left[\frac{43-3\sum_{t=1}^{r} 2^t}{2^r}\right]^2\left[\frac{43-3\sum_{t=1}^{r} 2^t}{2^r}+3\right]^2 4}\right]$$

$$= \frac{1}{(43)^2} - \frac{2^{14}}{\left[43-3\sum_{t=1}^{14} 2^t\right]} = -0.027264633.$$

The following example is the illustrates of Theorem 3.4.

Example 3.6. In (9), by taking $n = 1, k = 32, \ell = 3$ and $q = 2$, we get

$$\sum_{r=0}^{\infty}\left[\frac{-1}{2\left((32)2^r + 3\sum_{i=o}^{r} 2^i + 3\right)} - \frac{3}{\left((32)2^r + 3\sum_{i=o}^{r} 2^i + 3\right)^{(2)}_{3}}\right]$$

$$= -\frac{1}{32} \approx -0.03125.$$

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