SEMIGROUP IDEALS AND ADDITIVE MAPPINGS IN PRIME RINGS

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Abstract: The objective of this paper is to investigate commutativity of prime rings under appropriate additional conditions involving additive mappings and derivations. Moreover, examples proving the necessity of the primness hypothesis are given.

Keywords: Prime rings, Additive Mappings, Semi group Ideal, Commutators.

Introduction: The history of commuting and centralizing mapping goes back to 1955 when Divinsky [6] proved that a simple Artinien ring is commutative if it has a commuting non trivial automorphism. Two years later Posner[11] has proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner Second Theorem). In[8], the author obtained the same result for a centralizing automorphism and then these two results were generalized in [9] by showing that the ring is commutative if the automorphism or derivation centralizes and leaves invariant a nonzero ideal in the ring. This result subsequently redefined and extended by a number's of algebraists[1],[3],[5],[10] in several ways. In this note, we extend some result of Bell and Martindale [2] for a special type function and also, an example given to shows that the restriction imposed on the hypothesis of these results are superfluous.

The concept of generalized derivation is introduced by Bresar[4]. In[7], Hvala gave the algebraic study of generalized derivation of prime ring. Recall that *if* R *has the property* $Rx = \{0\} \Rightarrow x = 0$ *and* $f: R \to R$ *is an additive mapping such that*

$$f(xy) = f(x)y + xd(y) = d(x)y + xf(y),$$

for all $x, y \in R$. and d some function $d: R \to R$, then f uniquely determined by d. Moreover, d must be a derivation by ([4], Remark 1).

Throughout the paper, R will be represent an associative ring with multiplicative centre$Z(R)$. A ring R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$. A mapping $f: R \to R$ is called centralizing on I if $[f(x), x] \in Z(R)$ for all$x \in R$. In particular, it is called commuting on I if $[f(x), x] = 0$, for all $x \in R$.

In the following we state a well known fact as:

Remark 1.1 *For a nonzero element* $a \in Z(R)$ *such that* $ab \in Z(R)$, *then* $b \in Z(R)$.

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In all that follows, unless stated otherwise, R will be a prime ring and I a nonzero semigroup ideal of R. We denote $g: R \to R$ determined by a function h (not necessary additive) of R by (g, h) such that $g(xy) = g(x)y + xh(y)$, for all $x, y \in R$. In order to prove the main results we find it necessary to establish the following lemmas.

Lemma 2.1 *If* g *is centralizing on* I *of* R, *then* $g(x) \in Z(R)$, *for all* $x \in I \cap Z(R)$.

Proof. Since g is centralizing on I, we have

$[g(x), y] + [g(y), x] \in Z(R), \forall x, y \in I.$ (1)

Now, if $x \in Z(R)$, then we have

$$[g(x), y] \in Z(R), \forall x, y \in I.$$

The substitution $g(x)y$ for y in the above relation gives because of it

$$g(x)[g(x), y] \in Z(R), \forall x, y \in I.$$

If $[g(x), y] = 0$, then $g(x) \in C_R(I)$, the centralizer of I in R, and hence belong to $Z(R)$ by [2], Identity IV. But on the other hand, if $[g(x), y] \neq 0$, it again follows from Remark 1.1 that $g(x) \in Z(R)$.

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Lemma 2.2 *([4], Remark 1) Let* $g: R \to R$ *be an additive mapping and* h *be any mapping of* R *such that* $g(xy) = g(x)y + xh(y)$ *for all* $x, y \in R$. *If* R *is a semiprime ring, then* h *must be a derivation of* R.

Theorem 2.3 *If* g *is nonzero on* R *and commuting on* I, *then* R *is commutative.*

Proof. Since g is commuting on I, we have

$$[g(x), x] = 0, \forall x \in I.$$

The substitution $x + y$ for x in the last relation gives

$$[g(x), y] + [g(y), x] = 0, \forall x, y \in I.$$

Again, the substitution yx for y in the above relation gives us

$$yh(x)x = xyh(x), \forall x, y \in I.$$

Replacing y with ry in last identity, we have

$$[x, r]Ih(x) = 0, \forall x \in I, r \in R.$$

Since R is prime, therefore either $[x, r] = 0$ or $h(x) = 0$ for all $r \in R$. So, for any element $x \in I$ either $x \in Z(R)$ or $h(x) = 0$. Since g is nonzero on R then by Lemma 2.2, h is derivation in R. It follows from[10], Lemma 2 that h is nonzero on I. Suppose $h(x) \neq 0$ for some $x \in I$, then $x \in Z(R)$. Suppose $z \in I$ is such that $z \notin Z(R)$, then $h(z) = 0$ and $x + z \notin Z(R)$ this implies that $h(x + z) = 0$ and so $h(x) = 0$, a contradiction. This implies $z \in Z(R)$ for all $z \in I$. Thus I is commutative and hence, we conclude by ([10], Lemma 3) that R is commutative.

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Now we are in position to prove the result which involves centralizing mapping on I containing central elements.

Theorem 2.4 *Let g be nonzero on R and g is centralizing on I. If $I \cap Z(R) \neq 0$, then R is commutative*

Proof. We assume that $Z(R) \neq 0$ because otherwise g is commuting on I and there is nothing left to prove. Now, for a nonzerz $\in Z(R)$, we replace x by zy in ((1)) and we get

$[g(z), y]y + z[h(y), y] + z[g(y), y] \in Z(R),$

for all $y, z \in I$. It follows from Lemma 2.1 that

$$g(x) \in Z(R)$$

and there

$$z[h(y), y] + z[g(y), y] \in Z(R).$$

But as g is commuting on I, we have

$$z[g(y), y] \in Z(R),$$

and consequently,

$$z[h(y), y] \in Z(R).$$

Since z is nonzero, it follows from Remark 1.1 that $[h(y), y] \in Z(R)$. This implies that h centralizing on I. Hence, an application of Lemma 2.2 together with ([2], Theorem 4) gives the required result here. ◊

For completeness of we conclude this paper by giving an example to shows that the restrictions in the speculations of several results are not superfluous.

Example Let $R = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} | a, b \in S\right\}$ and $I = \left\{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} | b \in S\right\}$, where S is any ring. It is straight forward to see that R is not prime and I be a nonzero semigroup ideal of R. Now we define maps $g, h: R \to R$ by $g\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a+b & 0 \end{pmatrix}$ and $h\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix}$. It is easy to check that g, h satisfies all the requirement of Theorem 2.3 and Theorem 2.4 and however, R is not commutative. Hence, the hypothesis of primeness is crucial

Acknowledgement

The first author is thankful to the Department of Atomic Energy, Government of India, for its financial assistance provided through NBHM Postdoctoral Fellowship no. 2/40(14)/2014/ R&D-II/7794 to carry out this research work.

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