GENERALIZED GRAVITO-HEAVISIDIAN FIELD EQUATIONS WITH SPLIT OCTONIONS

B. C. CHANYAL

Abstract: In this paper, we describe the properties of split octonions and their connection with the 2x2 Zorn vector matrix containing both scalars and vectors using a modified version of matrix multiplication. We have written the generalized gravito-Heavisidian (GGH) potential, field and various wave equations of gravito-dyons in terms of split octonions. Accordingly, we demonstrate the work-energy theorem of classical mechanics reproducing the continuity equation for the case of gravito-dyons in terms of split octonions.

Keywords: octonions, split octonions, gravito-dyons, linear gravity, work-energy theorm.

Introduction: Octonions were first introduced in physics by Jordan, von Neumann and Wigner [1] who investigated a new finite Hilbert space, on replacing the complex numbers by octonions. Decomposition of four algebras, in view of celebrated Hurwitz theorem [2], has been characterized from Cayley Dickson process over the field of real numbers of dimensions n = 1, n = 2, n = 4 and n = 8 respectively for real, complex, quaternion and octonion algebras. So, there has been a revival in the formulation of natural laws so that there exists [2] four-division algebras consisting the algebra of real numbers (R), complex numbers (C), quaternions (H) and Octonions (O). All four algebra's are alternative with totally anti symmetric associators. Octonions [3] share with complex numbers and quaternions, many attractive mathematical properties, one might expect that they would be equally as useful as others. Octonion analysis has been widely discussed by Baez [4]. Sedenions, like quaternions or even octonions, form a 16- dimensional algebra over the reals, are used for unify the theory of gravi-electromagnetism [5].

On the other hand, the classical theory of gravity is, of course, Newton's law of universal gravitation. In 1893 Heaviside [6] investigated the analogy between gravitation and electromagnetism where he explained the propagation of energy in a gravitational field, in terms of a gravito-electromagnetic Poynting vector. Like magnetic field, Cantani [7] introduced a new field (called Heavisidian field) depending upon the velocities of gravitational charge (masses) and derived the covariant equations (like Maxwell's equations) of linear gravity.

Chanyal at el. studied [8]-[12] the octonion quantum chromodynamics, generalized octonion electrodynamics, generalized split-octonion electrodynamics, octonionic non-Abelian gauge theory, octonions and conservation laws for dyons and obtained the corresponding field equations (Maxwell's equations) and other quantum equation in compact and simpler formulation. In the present paper, we describe the properties of split octonions and their connection with the 2_2 Zorn vector matrix containing both scalars and vectors using a modified version of matrix multiplication. Thus, we have written the generalized gravito-Heavisidian (GH) potential, field and current equations of gravito-dyons in terms of split octonions. Accordingly, we have obtained the gravitational analogous of Generalized Dirac-Maxwell's equations, Proca-Maxwell's equations and the continuity equations. Furthermore, we have discussed the work energy theorem for gravito-dyons in the case of linearized gravitational theory.

Preliminaries: An octonion $\xi \in O$ is expressed as a real linear combination of the unit octonions as

$$\xi = \xi_0 e_0 + \sum_{j=1}^{7} \xi_j e_j \ , \tag{1}$$

where e_j $(j = 1,2,3,\ldots.7)$ are imaginary octonion units and eo is the real octonion unit element. The octonion unit element satisfy the following properties

$$\begin{aligned} & e_0 = 1, \ e_j^2 = -1, \\ & e_j e_k = -\delta_{jk} e_0 + f_{jkl} e_l , \ \ (j,k,l = 1,2,\ldots.7) \\ & [e_j , e_k] = 2 f_{jkl} e_l , \ \ \{e_j , e_k\} = -\delta_{jk} e_0 \ , \end{aligned} \tag{2}$$

where the structure constants f_{jkl} are completely antisymmetric and take the value 1 for $jkl =$ (123); (471); (257); (165); (624); (543); (736) and δ_{jk} is the usual Kronecker delta-Dirac symbol.

The split octonions algebra (9) are visualized the set of basis elements u_0, u_0^*, u_j, u_j^* $(j = 1,2,3)$ as

$$\begin{aligned} & u_0 = \frac{1}{2}(e_0 + ie_7), \ u_j = \frac{1}{2}(e_j + ie_{j+3}), \\ & u^*{}_0 = \frac{1}{2}(e_0 - ie_7), \ u^*{}_j = \frac{1}{2}(e_j - ie_{j+3}), \end{aligned} \tag{3}$$

where $i = \sqrt{-1}$ is assumed to commute with e_j $(j = 1,2,\ldots 7)$ octonion units. The split octonion basis elements satisfy the following multiplication rules,

$$u_i u_j = \varepsilon_{ijk} u_k^*, \; u_i^* u_j^* = -\varepsilon_{ijk} u_k^*, (\forall i, j, k = 1,2,3)$$
$$u_i u_j^* = \delta_{ij} u_0, \; u_i u_0 = 0, \; u_i^* u_0 = u_i^*,$$
$$u_0^* u_i^* = u_i, \; u_0^2 = u_0, \; u_i u_0^* = u_i, \; u_i^* u_0^* = 0. \tag{4}$$

We may now introduce a convenient realization for the basis elements (u_0, u_0^*, u_j, u_j^*) in term of Pauli's spin matrices as

$$u_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \; u_0^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$
$$u_j = \begin{pmatrix} 0 & 0 \\ e_j & 0 \end{pmatrix}; \; u_j^* = \begin{pmatrix} 0 & -e_j \\ 0 & 0 \end{pmatrix}. \tag{5}$$

The hermitian conjugation for split octonion basis elements can be defined in terms of both complex and octonion conjugation as

$$u_j^\dagger = \bar{u}_j^* = -u_j^*, \; u_0^\dagger = \bar{u}_0^* = u_0. \tag{6}$$

Split octonions and Zorn's vector-matrix representation: The split octonions are non-associative they cannot be represented by ordinary matrices because the matrix multiplication is always associative. Zorn found a new idea to represent them in 2x2 matrices containing both scalars and vectors using a modified version of matrix multiplication. Thus, the Zorn's vector-matrix and its determinant may be defined as

$$Z \to \left\{ \begin{pmatrix} m & \vec{p} \\ \vec{q} & n \end{pmatrix}; \; m, n \in R; \; \vec{p}, \vec{q} \in R^3 \right\} \tag{7}$$

And

$$|Z| \to \det \begin{pmatrix} m & \vec{p} \\ \vec{q} & n \end{pmatrix} = mn - \vec{p}.\vec{q}\,, \tag{8}$$

where m, n and $\vec{p}, \vec{q}$ are the scalar and vector coefficients. The product of two Zorn's vector matrices is expressed as

$$\begin{pmatrix} a & \vec{x} \\ \vec{y} & b \end{pmatrix} * \begin{pmatrix} c & \vec{u} \\ \vec{v} & d \end{pmatrix} = \begin{pmatrix} a\,c + (\vec{x}.\vec{v}) & a\,\vec{u} + d\,\vec{x} + (\vec{y} \times \vec{v}) \\ c\,\vec{y} + b\,\vec{v} - (\vec{x} \times \vec{u}) & b\,d + (\vec{y}.\vec{v}) \end{pmatrix}, \tag{9}$$

where (.) and $(\times)$ are the ordinary dot product and cross product of three-vectors.

The determinant is a quadratic form on the Zorn's algebra [13] which satisfies the following property:

$$\det[AB] = \det[A]\det[B], \; \forall \, (A, B) \in Z \tag{10}$$

Since, the Zorn's vector-matrix algebra is, in fact, isomorphic to the algebra of split-octonions (O). So, we can expressed the split octonion algebra in terms of 2x2 Zorn's vector matrices components of which are scalar and vector parts of a quaternion as

$$O \to \left\{ \begin{pmatrix} r & \vec{v} \\ \vec{w} & s \end{pmatrix}; \; r, s \in Sc(H); \; \vec{v}, \vec{w} \in Vce(H) \right\} \tag{11}$$

As such, we may write an arbitrary split octonion $\zeta \in O$ and its octonion conjugate $\bar{\zeta}$ in terms of following 2x2 Zorn's vector matrix realization as

$$\zeta = a u_0^* + b u_0 + x_i u_i^* + y_i u_i \to \begin{pmatrix} a & -\vec{x} \\ \vec{y} & b \end{pmatrix}, \tag{12}$$

$$\bar{\zeta} = a u_0 + b u_0^* - x_i u_i^* - y_i u_i \to \begin{pmatrix} a & \vec{x} \\ -\vec{y} & b \end{pmatrix}, \tag{13}$$

where *a* and *b* are scalars while $\vec{x}$ and $\vec{y}$ are three-vectors. The norm of ζ is defined as

$$N(\zeta) = \bar{\zeta}\zeta = \zeta\bar{\zeta} = (a\,b + \vec{x}.\vec{y})\hat{1} = n(\zeta)\hat{1}, \tag{14}$$

where $\hat{1}$ is the identity elements of matrix order 2x2, and the expression $n(\zeta) = (a\,b + \vec{x}.\vec{y})$.

Linearized GH-field equations: The split octonion differential operator *D* may be written in terms of scalor (diagonal coefficients) and vector part (off-diagonal coefficients) of 2x2 Zorn's vector matrix realization as [12]

$$D = \begin{pmatrix} \partial_t & -\vec{\nabla} \\ \vec{\nabla} & -\partial_t \end{pmatrix}, \tag{15}$$

where $\partial_t \to \dfrac{\partial}{\partial t}$ is the scalar component while $\vec{\nabla}$ is the vector component of Zorn's vector matrix. The idea of Dowker and Roche [14] of dual mass playing the role of magnetic charge (Heavisidian monopole), the gravito-dyons (also called g-dyons) may be defined as the particle carrying gravitational mass *m* and dual mass (Heavisidian mass) *h* having the effective mass of gravito-dyons is defined [15] as

$$M_{eff} = (m + h). \tag{16}$$

Analogous to the theory of electromagnetic of dyons, the generalized gravito-Heavisidian potential *U* in terms of 2x2 Zorn's vector matrix components of split octonions are expressed [9] as

$$U = \begin{pmatrix} \phi_H - \phi_G & -(\vec{A} + \vec{B}) \\ (\vec{A} - \vec{B}) & \phi_H + \phi_G \end{pmatrix}, \tag{17}$$

where $(\phi_G, \vec{A}) = \{A^\mu\}$ the potential is the gravitational four-vector potential due to the presence of gravitational charge (mass) and $(\phi_H, \vec{B}) = \{B^\mu\}$ is the Heavisidian four potential due to the existence of Heavisidian monopole. As such, the generalized gravito- Heavisidian (GH) vector field of Ψ gravito dyons can be expressed as

$$\overline{D}U = \Psi \Rightarrow \begin{pmatrix} 0 & -\left(\vec{H}+\vec{G}\right) \\ \left(\vec{H}-\vec{G}\right) & 0 \end{pmatrix}, \tag{18}$$

where $\vec{G}$ denotes the linear gravitational field (G-field) and $\vec{H}$ is the Heavisidian field (H-field) of gravito-dyons.

We have used the unit value of coefficients along with natural units $(c = \hbar = 1)$ and also take unit gravitational constant throughout the text. Thus, the generalized linear gravitational and Heavisidian fields of gravito-dyons are now expressed in terms of components of two four potentials [8, 9] in a symmetrical manner i.e.

$$\begin{aligned} \vec{G} &= -\vec{\nabla}\phi_G - \frac{\partial \vec{A}}{\partial t} - \left(\vec{\nabla}\times\vec{B}\right), \\ \vec{H} &= -\vec{\nabla}\phi_H - \frac{\partial \vec{B}}{\partial t} + \left(\vec{\nabla}\times\vec{A}\right). \end{aligned} \tag{19}$$

Thus, we may obtain the following expression in terms of 2x2 Zorn's vector matrix realization of split octonion for generalized GH-field of gravito-dyons as

$$D\,\Psi = -J\,, \tag{20}$$

where J is the split octonion equivalent of generalized four-current, can be expressed in the form of 2x2 Zorn's vector matrix as

$$J \Rightarrow \begin{pmatrix} \rho_H - \rho_G & -\left(\vec{J}+\vec{K}\right) \\ \left(\vec{J}-\vec{K}\right) & \rho_H + \rho_G \end{pmatrix}. \tag{21}$$

Where

$\left(\rho_G, \vec{J}\right) = \{J^\mu\}$, *and* $\left(\rho_H, \vec{K}\right) = \{K^\mu\}$,

are respectively the four currents associated with the gravitational mass and Heavisidian monopole for the case of generalized GH-field of gravito-dyons. Thus, equation (20) leads to the following linear GH-field equations in flat space as

$$\begin{aligned} \left(\vec{\nabla}.\vec{G}\right) &= -\rho_G\,, \\ \left(\vec{\nabla}\times\vec{G}\right) &= -\frac{\partial \vec{H}}{\partial t} + \vec{K}\,, \\ \left(\vec{\nabla}\times\vec{H}\right) &= \frac{\partial \vec{G}}{\partial t} - \vec{J}\,, \\ \left(\vec{\nabla}.\vec{H}\right) &= -\rho_H\,, \end{aligned} \tag{22}$$

which are the gravitational analogous of Generalized Dirac-Maxwell's (GDM) equations for linear GH-field of gravito-dyons and allow for the possibility of Heavisidian monopole and currents, analogous to gravitational charges (masses) and currents.

Here, the gravitational analogue of GDM equations (22) are invariant not only under Lorentz and conformal transformations but also under the duality transformations [12, 13]. On the other hand, the expression for the generalized octonion GH-field of massive gravito-dyons will be [14],

$$D\,\Psi - M^2\,U = -J\,, \tag{23}$$

where $M \approx M_{eff}$ is the effective mass of gravito-dyons given in equation (16). So, the generalized split octonion GH-field of massive gravito-dyons lead to following differential equations in the form of compact notation

$$\begin{aligned} \left(\vec{\nabla}.\vec{G}\right) + M^2\,\phi_G &= -\rho_G\,, \\ \left(\vec{\nabla}\times\vec{G}\right) + M^2\,\vec{A} &= -\frac{\partial \vec{H}}{\partial t} + \vec{K}\,, \\ \left(\vec{\nabla}\times\vec{H}\right) + M^2\,\vec{B} &= \frac{\partial \vec{G}}{\partial t} - \vec{J}\,, \\ \left(\vec{\nabla}.\vec{H}\right) + M^2\,\phi_H &= -\rho_H\,, \end{aligned} \tag{24}$$

which are the generalized Proca-Maxwell's (GPM) like equations for the case of GH-field of massive gravito-dyons.

Linear GH-wave equations and Conservation laws: In order to write the split-octonion form of potential wave equations for gravito-dyons, we start from the following GH-field equation,

$$D\,\overline{D}\,U = \overline{D}\,D\,U = -J\,, \tag{25}$$

and finally we obtain the following form of wave equations

$$\begin{aligned} \nabla^2\,\phi_G - \frac{\partial^2 \phi_G}{\partial t^2} &= -\rho_G\,, \\ \nabla^2\,\phi_H - \frac{\partial^2 \phi_H}{\partial t^2} &= -\rho_H\,, \\ \nabla^2\,\vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} &= -\vec{J}\,, \\ \nabla^2\,\vec{B} - \frac{\partial^2 \vec{B}}{\partial t^2} &= -\vec{K}\,. \end{aligned} \tag{26}$$

On the other hand, using the octonionic Proca equation [14], we obtain the following wave equations for the case of massive gravito-dyons, i.e.

$$\begin{aligned} \left(\nabla^2 - M^2\right)\phi_G - \frac{\partial^2 \phi_G}{\partial t^2} &= -\rho_G\,, \\ \left(\nabla^2 - M^2\right)\phi_H - \frac{\partial^2 \phi_H}{\partial t^2} &= -\rho_H\,, \\ \left(\nabla^2 - M^2\right)\vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} &= -\vec{J}\,, \\ \left(\nabla^2 - M^2\right)\vec{B} - \frac{\partial^2 \vec{B}}{\partial t^2} &= -\vec{K}\,. \end{aligned} \tag{27}$$

which are the potential wave equations for generalized GH-field of gravito-dyons. Now, if we may operate $\overline{\Psi}$ as:

$$\overline{\Psi}\left(D\,\Psi\right)=-\overline{\Psi}J\,. \tag{28}$$

So, we obtain the scalar coefficient of (28) as

$$\frac{1}{2}\frac{\partial}{\partial t}(G^2+H^2)+\vec{\nabla}.(\vec{G}\times\vec{H})+(\vec{H}.\vec{K}+\vec{G}.\vec{J})=0 \tag{29}$$

which represents the work-energy theorem or Poynting theorem for gravito-dyons. In this case, the energy stored due to the generalized GH-field of gravito-dyons and the Poynting vector can be defined as

$$W_{GH}=\frac{1}{2}\int(G^2+H^2)\,d\tau,\quad S_{GH}=(\vec{G}\times\vec{H}). \tag{30}$$

Furthermore, the Poynting theorem also may be generalized as the conservation of energy for gravito-dyonic as

$$\frac{\partial W}{\partial t}=-\frac{\partial W_{GH}}{\partial t}-\vec{\nabla}.\vec{S}_{GH}-(\vec{H}.\vec{K}+\vec{G}.\vec{J}). \tag{31}$$

Moreover, in order to write the conservation law of momentum for gravito-dyons, we equate the vector coefficient of (28), and obtain the following relation,

$$\begin{aligned}&\frac{\partial\vec{S}_{GH}}{\partial t}+\frac{1}{2}\vec{\nabla}(G^2+H^2)-(\vec{H}.\vec{\nabla})\vec{H}-(\vec{G}.\vec{\nabla})\vec{G}\\&-\vec{H}\left(\vec{\nabla}.\vec{H}\right)-\vec{G}\left(\vec{\nabla}.\vec{G}\right)\\&=\rho_H\vec{H}+\rho_G\vec{G}-(\vec{H}\times.\vec{J})+(\vec{G}\times\vec{K})\end{aligned} \tag{32}$$

which gives rise the connection between generalized gravito-Heavisidian (GH) energy and the force due to presence of linear gravitational and Heavisidian energy of gravito-dyons.

Conclusion: We have discussed the gravito-Heavisidian field (GH-field) of gravito-dyons and obtained the gravitational analogous of GDM equations, GPM equations and other wave equations in the case of 2x2 Zorn's vector matrix realization of split-octonionic form of linear gravitational space. We have described the gravitational analogous of gravito-Heavisidian (GH) work-energy theorem or Poynting theorem and the conservation of linear momentum for gravito dyons for the case of split-octonion. The advantage of the present formulation of gravito-dyons in split octonion formalism is that in this form one can extend the theory of gravitation into the curved spaces instead of flat spaces then we can discuss the existence of gravitational waves also.

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B. C. Chanyal/Department of Physics/G. B. Pant University of Agri. & Tech./Pantnagar/Uttarakhand/India/